①

Exam 2 review:

To solve a single differential equation, for exam 2, use Ch 5 methods: $x=0$ regular singular pt

A.) If you have an Euler equation, $x^2y'' + \alpha x y' + \beta y = 0$ where α, β are constants, use simple 5.4 method (guess $y = |x|^r$, breaks into standard 3 cases, see 5.4 handouts).

B.) Suppose you are interested in the solution near $x = x_0$, then we can find

(1.) exact solution solution by solving for the series solution (ex: see 5.2 handout)

(2.) An approximate solution by determining the first few terms in the series solution (ex: see 5.5 part 2 handout)

Determine if x_0 is an ordinary point, regular singular value, or irregular singular value.

If x_0 is an ordinary point, solution near x_0 is $\Sigma_{n=0}^{\infty} a_n (x - x_0)^n$. $-x_0 = 0$

If x_0 is a regular singular point, solution near x_0 is $\Sigma_{n=0}^{\infty} a_n (x - x_0)^{n+r}$.

$x_0 = 0$

When (and where) do you know when solution exists?

What are the subparts of these problems?

Look at theory including existence, uniqueness, domain of solution, linearity.

IVP

domain need not include x_0
Example: Euler's eqn
$r = 1/2$

To solve a system of differential equations use Ch 7 methods:

Linear: find eigenvalues, eigenvectors, breaks into standard 3 cases (plus a subcase) – see last 7.5 handout

When do you know a solution exists? uniqueness? Linearity properties?

Be able to translate an *nth* order linear differential equation into a system of n linear differential equations and write in matrix form.

Understand and be able to identify different types of critical points (equilibrium solutions = constant solutions) for both linear and non-linear systems.

* asymptotically ~~stable~~, stable, unstable
* sink, center, source
* spiral, node, saddle

stable
center
r is imaginary

Be able to graph phase portrait of a linear system of DE (trajectories in x_1, x_2-plane). Also be able to graph x_i versus t for simple cases.

Completely understand Fig 9.1.9.

Look at theory including existence, uniqueness, domain of solution, linearity.

domain
includes x_0

node
saddle
real case

spiral
complex
not imaginary
not real
$a+bi$

Ex: $y = c_1|x|^{1/2} + c_2 x^3$

Domain is at least as large as $(-\infty, 0)$ or $(0, \infty)$

2

Solve $x^2y'' + \alpha xy' + \beta y = 0$. Let $y = x^r$,
$y' = rx^{r-1}$, $y'' = r(r-1)x^{r-2}$ (case when $y = (-x)^r$ is similar).

$$x^2x^{r-2}r(r-1) + \alpha x x^{r-1}r + \beta x^r = 0$$

$x^r[r^2 - r + \alpha r + \beta] = 0$ for all x implies $r^2 + (\alpha - 1)r + \beta = 0$

Thus x^r is a solution iff $r = \frac{-(\alpha-1)\pm\sqrt{(\alpha-1)^2-4\beta}}{2}$

Case 1: Two real roots, r_1, r_2.

General solution is $y = c_1|x|^{r_1} + c_2|x|^{r_2}$

remove $| \ |$ if x^{r_i} defined

Case 2: Two complex roots, $r_i = \lambda \pm i\mu$:

Convert solution to form without complex numbers.

Note $|x|^{\pm i\mu} = e^{ln(|x|^{\pm i\mu})} = e^{(\pm i\mu)ln|x|} = e^{i(\pm \mu ln|x|)}$

$$= cos(\pm\mu ln|x|) + isin(\pm\mu ln|x|)$$
$$= cos(\mu ln|x|) \pm isin(\mu ln|x|)$$

General solution is $y = c_1|x|^{r_1} + c_2|x|^{r_2} = c_1|x|^{\lambda+i\mu} + c_2|x|^{\lambda-i\mu}$

$= |x|^\lambda(c_1|x|^{i\mu} + c_2|x|^{-i\mu})$

$= |x|^\lambda(c_1[cos(\mu ln|x|) + isin(\mu ln|x|)] + c_2[cos(\mu ln|x|) - isin(\mu ln|x|)])$

$= |x|^\lambda([c_1 + c_2]cos(\mu ln|x|) + i[c_1 - c_2]sin(\mu ln|x|))$

$= |x|^\lambda(k_1 cos(\mu ln|x|) + k_2 sin(\mu ln|x|))$

$= k_1|x|^\lambda cos(\mu ln|x|) + k_2|x|^\lambda sin(\mu ln|x|)$

Case 3: one repeated root, $r_1 = \frac{-(\alpha-1)}{2}$. (i.e., $\sqrt{(\alpha-1)^2 - 4\beta} = 0$):

Thus $|x|^{r_1}$ is a solution. Find 2nd solution.

Method 1. Reduction of order: Suppose $y = u(x)|x|^{r_1}$ is a solution to $x^2y'' + \alpha xy' + \beta y = 0$. Plug in and determine $u(x)$

Method 2: Let $L(y) = x^2y'' + \alpha xy' + \beta y$ where $y' = \frac{dy}{dx}$.

$L(|x|^r) = |x|^r(r - r_1)^2$

$\frac{\partial}{\partial r}[L(|x|^r)] = \frac{\partial}{\partial r}[|x|^r(r-r_1)^2] = (|x|^r)'(r-r_1)^2 + 2|x|^r(r-r_1) = 0$ if $r = r_1$.

Suppose x is constant with respect to r and all the partial derivatives are continuous. Then

$$\frac{\partial}{\partial r}[L(y)] = \frac{\partial}{\partial r}[x^2y'' + \alpha xy' + \beta y] = x^2\frac{\partial y''}{\partial r} + \alpha x\frac{\partial y'}{\partial r} + \beta\frac{\partial y}{\partial r}$$
$$= x^2\frac{\partial}{\partial r}[\frac{\partial^2 y}{\partial x^2}] + \alpha x\frac{\partial}{\partial r}[\frac{\partial y}{\partial x}] + \beta\frac{\partial y}{\partial r}$$
$$= x^2\frac{\partial^2}{\partial x^2}[\frac{\partial y}{\partial r}] + \alpha x\frac{\partial}{\partial x}[\frac{\partial y}{\partial r}] + \beta\frac{\partial y}{\partial r}$$
$$= L(\frac{\partial y}{\partial r}) \text{ for all } r$$

$L(\frac{\partial |x|^r}{\partial r}) = \frac{\partial}{\partial r}[L(|x|^r)] = 0$ for $r = r_1$.

$\frac{\partial|x|^r}{\partial r} = \frac{\partial e^{ln|x|^r}}{\partial r} = \frac{\partial e^{rln|x|}}{\partial r} = (e^{rln|x|})ln|x| = |x|^r ln|x|$

Thus $|x|^{r_1}ln|x|$ is a solution.

Thus general solution is $y = c_1|x|^{r_1} + c_2|x|^{r_1}ln|x|$

since by the Wronskian, $|x|^{r_1}$ and $|x|^{r_1}ln|x|$ are linearly independent.
Suppose $x > 0$ and $r_1 \neq 0$.

$$\begin{vmatrix} x^{r_1} & x^{r_1}ln|x| \\ r_1x^{r_1-1} & r_1x^{r_1-1}ln|x| + x^{r_1-1} \end{vmatrix}$$ ← coeff matrix for finding c_1, c_2

$$= x^{r_1}(r_1x^{r_1-1}ln|x| + x^{r_1-1}) - x^{r_1}ln|x|r_1x^{r_1-1}$$

$$= x^{2r_1-1}[r_1ln|x| + 1 - ln|x|r_1] = x^{2r_1-1} \neq 0 \text{ for } x \neq 0$$

Other cases for Wronskian are similar.

$\Rightarrow \exists!$ sol to IVP

3 2

Solve $y'' - 4y' + 4y = 0$, $y(0) = y_0$

Using quick 3.4 method. Guess $y = e^{rt}$ and plug into equation to find $r^2 - 4r + 4 = 0$. Thus $(r-2)^2 = 0$. Hence $r = 2$. Therefore general solution is $y = c_1 e^{2x} + c_2 x e^{2x}$.

Use LONG 5.2 method (normally use this method only when other shorter methods don't exist) to find solution for values near $x_0 = 0$.

Suppose the solution $y = f(x)$ is analytic at $x_0 = 0$.

That is $f(x) = \Sigma_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n$ for x near $x_0 = 0$.

Thus there are constants $a_n = \frac{f^{(n)}(0)}{n!}$ such that,

$$\boxed{f(x) = \Sigma_{n=0}^{\infty} a_n (x-0)^n = \Sigma_{n=0}^{\infty} a_n x^n.} \Leftarrow \text{plug in}$$

Find a recursive formula for the constants of the series solution to $y'' - 4y' + 4y = 0$ near $x_0 = 0$

We will determine these constants a_n by plugging f into the ODE.

$f(x) = \Sigma_{n=0}^{\infty} a_n x^n$, $f'(x) = \Sigma_{n=1}^{\infty} a_n n x^{n-1}$, $f''(x) = \Sigma_{n=2}^{\infty} a_n n(n-1)x^{n-2}$.

$\Sigma_{n=2}^{\infty} a_n n(n-1)x^{n-2} - 4\Sigma_{n=1}^{\infty} a_n n x^{n-1} + 4\Sigma_{n=0}^{\infty} a_n x^n = 0$.

$\Sigma_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - 4\Sigma_{n=0}^{\infty} a_{n+1}(n+1)x^n + 4\Sigma_{n=0}^{\infty} a_n x^n = 0$.

$\Sigma_{n=0}^{\infty} [a_{n+2}(n+2)(n+1) - 4a_{n+1}(n+1) + 4a_n] x^n = 0$.

$a_{n+2}(n+2)(n+1) - 4a_{n+1}(n+1) + 4a_n = 0$.

$a_{n+2} = \frac{4a_{n+1}(n+1) - 4a_n}{(n+2)(n+1)}$.

Hence the recursive formula (if know previous terms, can determine later terms) is

$$a_{n+2} = 4\left(\frac{(n+1)a_{n+1} - a_n}{(n+2)(n+1)}\right)$$

Given the recursive formula, $a_{n+2} = 4\left(\frac{(n+1)a_{n+1} - a_n}{(n+2)(n+1)}\right)$, determine a_n.

Determine formula for a_k by noticing patterns. Note: It is easier to notice patterns if you do NOT simplify too much.

Find the first 6 terms of the series solution

$n = 0: \quad a_2 = 4\left(\frac{a_1 - a_0}{(2)(1)}\right)$

$n = 1: \quad a_3 = 4\left(\frac{2a_2 - a_1}{(3)(2)}\right) = 4\left(\frac{(2)(4)\left(\frac{a_1 - a_0}{(2)(1)}\right) - a_1}{(3)(2)}\right) = 4\left(\frac{4(a_1 - a_0) - a_1}{(3)(2)}\right)$

$= 4\left(\frac{3a_1 - 4a_0}{3!}\right)$

$n = 2: \quad a_4 = 4\left(\frac{3a_3 - a_2}{(4)(3)}\right) = 4\left(\frac{3(4)\left(\frac{3a_1 - 4a_0}{3!}\right) - 4\left(\frac{a_1 - a_0}{2!}\right)}{(4)(3)}\right) = 4\left(\frac{3\left(\frac{3a_1 - 4a_0}{3!}\right) - \left(\frac{a_1 - a_0}{2!}\right)}{3}\right)$

$= 4\left(\frac{\left(\frac{3a_1 - 4a_0}{2!}\right) - \left(\frac{a_1 - a_0}{2!}\right)}{3}\right) = 4\left(\frac{(3a_1 - 4a_0) - (a_1 - a_0)}{3!}\right) = 4\left(\frac{2a_1 - 3a_0}{(3!)}\right)$

$n = 3: \quad a_5 = 4\left(\frac{(4)a_4 - a_3}{(5)(4)}\right) = 4\left(\frac{(4)4\left(\frac{2a_1 - 3a_0}{3!}\right) - 4\left(\frac{3a_1 - 4a_0}{3!}\right)}{(5)(4)}\right)$

$= 4\left(\frac{4\left(\frac{2a_1 - 3a_0}{3!}\right) - \left(\frac{3a_1 - 4a_0}{3!}\right)}{5}\right) = 4\left(\frac{4(2a_1 - 3a_0) - (3a_1 - 4a_0)}{5(3!)}\right) = 4\left(\frac{5a_1 - 8a_0}{5(3!)}\right)$

$f(x) \sim a_0 + a_1 x + 4\left(\frac{a_1 - a_0}{2!}\right)x^2 + 4\left(\frac{3a_1 - 4a_0}{3!}\right)x^3 + 4\left(\frac{2a_1 - 3a_0}{(3!)}\right)x^4 + 4\left(\frac{5a_1 - 8a_0}{5(3!)}\right)x^5$

Recall $f(x) = a_0\phi_0(x) + a_1\phi_1(x)$ for linearly independent solutions ϕ_0 and ϕ_1 to equation $y'' - 4y' + 4y = 0$.

Find the first 5 terms in each of the 2 solns $y = \phi_0(x)$ and $y = \phi_1(x)$

$\phi_0 \sim 1 + 4\left(\frac{-1}{2!}\right)x^2 + 4\left(\frac{-4}{3!}\right)x^3 + 4\left(\frac{-3}{(3!)}\right)x^4 + 4\left(\frac{-8}{5(3!)}\right)x^5$

$\phi_1 \sim x + 4\left(\frac{1}{2!}\right)x^2 + 4\left(\frac{3}{3!}\right)x^3 + 4\left(\frac{2}{(3!)}\right)x^4 + 4\left(\frac{5}{5(3!)}\right)x^5$

$n = 0: \quad a_2 = 4\left(\frac{a_1 - a_0}{(2)(1)}\right) = 2\left(\frac{2a_1 - 2a_0}{2!}\right)$

$n = 1: \quad a_3 = = 4\left(\frac{3a_1 - 4a_0}{3!}\right) = 2^2\left(\frac{3a_1 - 4a_0}{3!}\right)$

$n = 2: \quad a_4 = 4\left(\frac{2a_1 - 3a_0}{3!}\right) = 16\left(\frac{2a_1 - 3a_0}{4!}\right) = 8\left(\frac{4a_1 - 6a_0}{4!}\right) = 2^3\left(\frac{4a_1 - 6a_0}{4!}\right)$

$n = 3: \quad a_5 = 4\left(\frac{5a_1 - 8a_0}{5(3!)}\right) = 16\left(\frac{5a_1 - 8a_0}{5!}\right) = 2^4\left(\frac{5a_1 - 8a_0}{5!}\right)$

Hence it appears $a_k = \frac{2^{k-1}(ka_1 - 2(k-1)a_0)}{k!}$

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In general, to determine if there is a unique solution to the IVP, $y'' - 4y' + 4y = 0$, $y(x_0) = y_0$, $y'(x_0) = y_1$, we solve for unknowns a_0 and a_1.

$$y(x_0) = a_0\phi_0(x_0) + a_1\phi_1(x_0)$$
$$y'(x_0) = a_0\phi_0'(x_0) + a_1\phi_1'(x_0)$$

Note that the above system of two equations has a unique solution for the two unknowns a_0 and a_1 if and only if $det \begin{pmatrix} \phi_0(x_0) & \phi_1(x_0) \\ \phi_0'(x_0) & \phi_1'(x_0) \end{pmatrix} \neq 0$

In other words the IVP has a unique solution iff the Wronskian of ϕ_0 and ϕ_1 evaluated at x_0 is not zero. Recall that by theorem , this also implies that ϕ_0 and ϕ_1 are linearly independent and hence the general solution is $y = a_0\phi_0(x) + a_1\phi_1(x)$ by theorem.

Show that $\phi_0(x) = (-2)\Sigma_{n=0}^{\infty}\frac{2^{n-1}((n-1))}{n!}x^n$ and $\phi_1(x) = \Sigma_{n=1}^{\infty}\frac{2^{n-1}}{(n-1)!}x^n$ are linearly independent by calculating the Wronskian of these two functions evaluated at $x_0 = 0$.

$$W(\phi_1,\phi_2)(x) = \begin{pmatrix} \phi_1(x) & \phi_2(x) \\ \phi_1'(x) & \phi_2'(x) \end{pmatrix} = \begin{pmatrix} (-2)\Sigma_{n=0}^{\infty}\frac{2^{n-1}(n-1)}{n!}x^n & \Sigma_{n=1}^{\infty}\frac{2^{n-1}}{(n-1)!}x^n \\ (-2)\Sigma_{n=1}^{\infty}\frac{2^{n-1}(n-1)}{(n-1)!x^{n-1}} & \Sigma_{n=1}^{\infty}\frac{n2^{n-1}}{(n-1)!}x^{n-1} \end{pmatrix}$$

$$W(\phi_1,\phi_2)(0) = \begin{pmatrix} (-2)2^{0-1}(-1) & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \neq 0$$

Hence $\phi_0(x) = (-2)\Sigma_{n=0}^{\infty}\frac{2^{n-1}((n-1))}{n!}x^n$ and $\phi_1(x) = \Sigma_{n=1}^{\infty}\frac{2^{n-1}}{(n-1)!}x^n$ are linearly independent

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When possible identify the functions giving the series solutions. Recall that by Taylor's theorem and the ratio test, $e^{2x} = \Sigma_{n=0}^{\infty}\frac{f^{(n)}(x)}{n!}x^n = \Sigma_{n=0}^{\infty}\frac{2^n}{n!}x^n$ for all x.

$$f(x) = a_1\Sigma_{n=0}^{\infty}\frac{n2^{n-1}}{n!}x^n - 2a_0\Sigma_{n=0}^{\infty}\frac{2^{n-1}(n-1)}{n!}x^n$$

$$= a_1\Sigma_{n=0}^{\infty}\frac{n2^{n-1}}{n!}x^n - 2a_0\Sigma_{n=0}^{\infty}\frac{n2^{n-1}}{n!}x^n + 2a_0\Sigma_{n=0}^{\infty}\frac{2^{n-1}}{n!}x^n$$

$$= (a_1 - 2a_0)\Sigma_{n=0}^{\infty}\frac{n2^{n-1}}{n!}x^n + a_0\Sigma_{n=0}^{\infty}\frac{2^n}{n!}x^n$$

$$= (a_1 - 2a_0)x\Sigma_{n=1}^{\infty}\frac{2^{n-1}}{(n-1)!}x^{n-1} + a_0\Sigma_{n=0}^{\infty}\frac{2^n}{n!}x^n$$

$$= (a_1 - 2a_0)x\Sigma_{n=0}^{\infty}\frac{2^n}{n!}x^n + a_0\Sigma_{n=0}^{\infty}\frac{2^n}{n!}x^n$$

$$= (a_1 - 2a_0)xe^{2x} + a_0e^{2x}$$

Note we have recovered the solution we found using the 3.4 method.

Note a power series solutions exists in a neighborhood of x_0 when the solution is analytic at x_0. I.e, the solution is of the form $y = \Sigma_{n=0}^{\infty}a_n(x - x_0)^n$ where this series has a nonzero radius of convergence about x_0.

When do we know an analytic solution exists? I.e, when is this method guaranteed to work?

Special case: $P(x)y'' + Q(x)y' + R(x)y = 0$

Then $y''(x) = -\frac{Q}{P}y' - \frac{R}{P}y$

Definition: The point x_0 is an *ordinary point* of the ODE,

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

if $\frac{Q}{P}$ and $\frac{R}{P}$ are analytic at x_0.

Theorem 5.3.1: If x_0 is an ordinary point of the ODE $P(x)y'' + Q(x)y' + R(x)y = 0$, then the general solution to this ODE is

$$y = \Sigma_{n=1}^{\infty}a_n(x - x_0)^n = a_0\phi_0(x) + a_1\phi_1(x)$$

where ϕ_i are power series solutions that are analytic at x_0. The solutions ϕ_0, ϕ_1 form a fundamental set of solutions. The radius of convergence for each of these series solutions is at least as large as the minimum radii of convergence of the series for $\frac{Q}{P}$ and $\frac{R}{P}$.

Theorem: If P and Q are polynomial functions, then $y = Q(x)/P(x)$ is analytic at x_0 if and only if $P(x_0) \neq 0$. Moreover if Q/P is reduced, the radius of convergence of $Q(x)/P(x) = min\{||x_0 - x|| \mid x \in \mathbf{C}, P(x) = 0\}$ where $||x_0 - x||$ = distance from x_0 to x in the complex plane.

use to find domain $(x_0 - r, x_0 + r)$

x_0 ordinary pt, can solve by translat. $x_0 \to 0$ & plugging in $y = \Sigma a_n x^n$

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5.5 Series Solutions Near a Regular Singular Point, Part I

Theorem 5.3.1: If $p(x)$ and$q(x)$ are analytic at x_0 (i.e., x_0 is an ordinary point of the ODE $y''+p(x)y'+q(x)y=0$), then the general solution to this ODE is

$$y = \Sigma_{n=0}^{\infty} a_n(x-x_0)^n = a_0\phi_0(x) + a_1\phi_1(x)$$

where ϕ_i are power series solutions that are analytic at x_0. The solutions ϕ_0, ϕ_1 form a fundamental set of solutions. The radius of convergence for each of these series solutions is at least as large as the minimum radii of convergence of the series for $\frac{Q}{P}$ and $\frac{R}{P}$.

If you prefer a power series expansion about 0, use u-substitution: let $u = x - x_0$. Then $p(u+x_0)$ and $q(u+x_0)$ are analytic at 0

(Semi-failed) attempt to transform 5.5 problem into 5.4 problem:

5.5: $y'' + p(x)y' + q(x)y = 0$

$x^2y'' + x^2p(x)y' + x^2q(x)y = 0$

$x^2y'' + x[xp(x)]y' + [x^2q(x)]y = 0$ where $xp(x)$ and $x^2q(x)$ are functions of x.

5.4: $x^2y'' + \alpha xy' + \beta y = 0$ **where** α, β **are constants.**

Combine 5.3/5.4 methods.

Defn: x_0 is a *regular singular value* if x_0 is a singular value and $xp(x)$ and $x^2q(x)$ are analytic at x_0. A singular value which is not regular is called *irregular*.

Examples:

$y'' + \frac{y'}{x} + \frac{y}{x^2} = 0$, regular singular value: $x = 0$.

$y'' + \frac{y'}{x^2} + \frac{y}{x} = 0$, irregular singular value: $x = 0$.

$y'' + y' + \frac{y}{x^3} = 0$, irregular singular value: $x = 0$.

If $p(x)$ and $q(x)$ are rational functions, then $xp(x)$ and $x^2q(x)$ are analytic iff $lim_{x\to 0}xp(x)$ and iff $lim_{x\to 0}x^2q(x)$ are finite. (i.e., after reducing fractions, x is not in the denominator.

Ex: $p(x) = \frac{1}{x}$ implies $xp(x) = \frac{x}{x} = 1$

Ex: $p(x) = \frac{1}{x^2}$ implies $xp(x) = \frac{x}{x^2} = \frac{1}{x}$

If $x_0 = 0$ is a regular singular value of the linear homogeneous DE, $x^2y'' + x[xp(x)]y' + x^2q(x)y = 0$ (*), then

$xp(x) = \Sigma_{n=0}^{\infty} p_nx^n$ and $x^2q(x) = \Sigma_{n=0}^{\infty} q_nx^n$ for constants p_n, q_n.

If $y = x^r\Sigma_{n=0}^{\infty} a_nx^n = \Sigma_{n=0}^{\infty} a_nx^{n+r}$ is a solution to (*) where $r \neq 0$.

$y' = \Sigma_{n=0}^{\infty}(n+r)a_nx^{n+r-1}$ and $y'' = \Sigma_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2}$

$$x^2\Sigma_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2} + x[xp(x)]\Sigma_{n=0}^{\infty}(n+r)a_nx^{n+r-1} + [x^2q(x)]\Sigma_{n=0}^{\infty}a_nx^{n+r}$$

$$\Sigma_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r} + [xp(x)]\Sigma_{n=0}^{\infty}(n+r)a_nx^{n+r} + [x^2q(x)]\Sigma_{n=0}^{\infty}a_nx^{n+r}$$

$$\Sigma_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r} + \left(\Sigma_{n=0}^{\infty}p_nx^n\right)\left(\Sigma_{n=0}^{\infty}(n+r)a_nx^{n+r}\right) + \left(\Sigma_{n=0}^{\infty}q_nx^n\right)\left(\Sigma_{n=0}^{\infty}a_nx^{n+r}\right)$$

Thus the coefficient of x^r is $r(r-1)a_0 + p_0ra_0 + q_0a_0 = 0$

We can take $a_0 \neq 0$. Thus $r(r-1) + p_0r + q_0 = 0$

Thus we can solve for r using the quadratic formula.

Case 1: $r_1 > r_2$ both real and $r_1 - r_2$ is not an integer.
Case 2: $r_1 > r_2$ both real and $r_1 - r_2 = p$, p an integer.
Case 3: one repeated root.
Case 4: two complex roots.

coefficient matrix of the given system. Show that the critical point (0, 0) is a

(a) Node if $q > 0$ and $\Delta \geq 0$; (b) Saddle point if $q < 0$;

(c) Spiral point if $p \neq 0$ and $\Delta < 0$; (d) Center if $p = 0$ and $q > 0$.

Hint: These conclusions can be reached by studying the eigenvalues r_1 and r_2. It may also be helpful to establish, and then to use, the relations $r_1 r_2 = q$ and $r_1 + r_2 = p$.

21. Continuing Problem 20, show that the critical point (0, 0) is

(a) Asymptotically stable if $q > 0$ and $p < 0$;

(b) Stable if $q > 0$ and $p = 0$;

(c) Unstable if $q < 0$ or $p > 0$.

$r = 0$ purely imag

The results of Problems 20 and 21 are summarized visually in Figure 9.1.9.

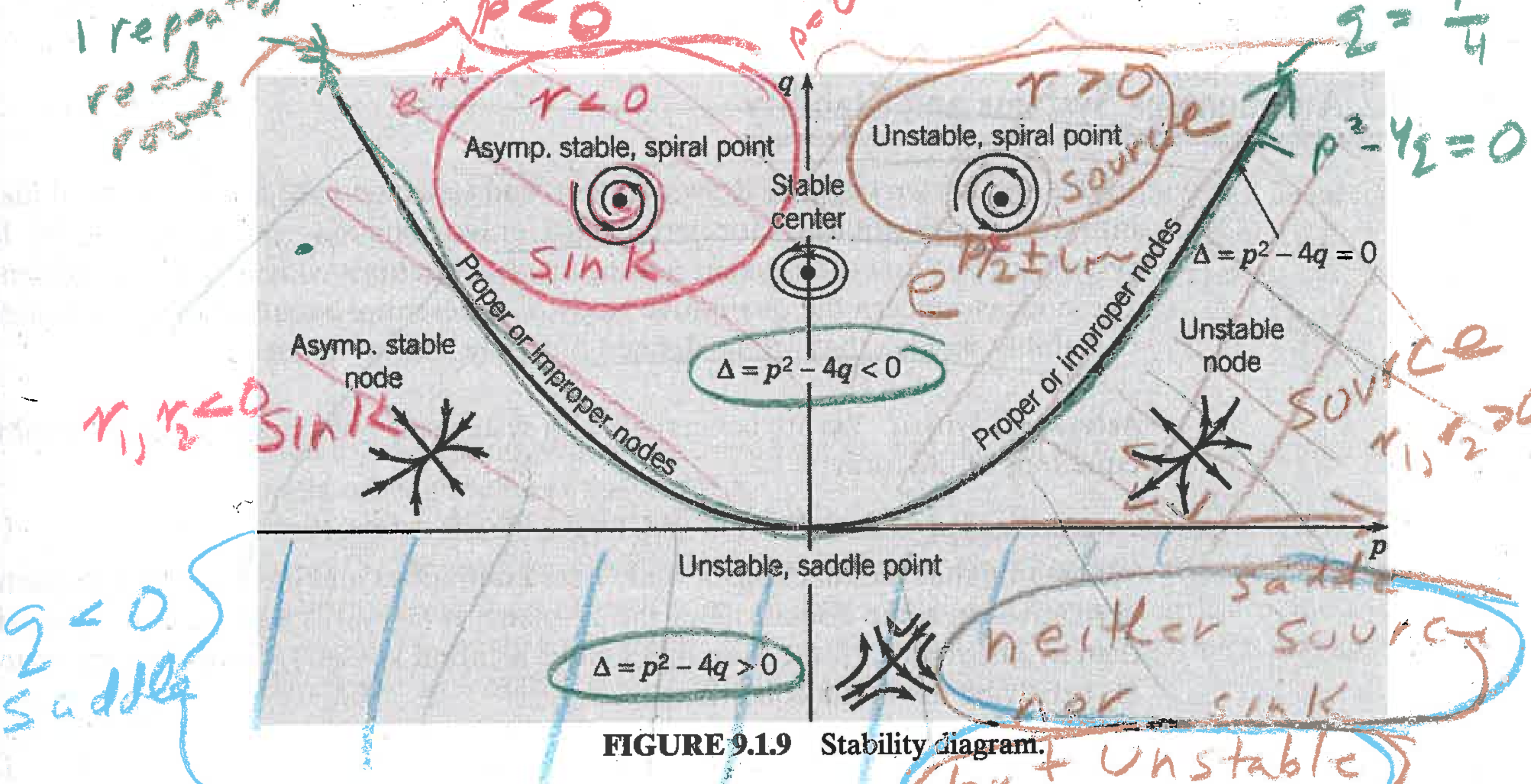

FIGURE 9.1.9 Stability diagram.

22. In this problem we illustrate how a 2×2 system with eigenvalues $\lambda \pm i\mu$ can be

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$p = a + d$

$q = ad - bc$

Ch 7 and 9

Suppose an object moves in the 2D plane (the x_1, x_2 plane) so that it is at the point $(x_1(t), x_2(t))$ at time t. Suppose the object's velocity is given by

$$x_1'(t) = ax_1 + bx_2,$$
$$x_2'(t) = cx_1 + dx_2$$

Or in matrix form $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

To solve, find eigenvalues and corresponding eigenvectors:

$$\begin{vmatrix} a-r & b \\ c & d-r \end{vmatrix} = (a-r)(d-r) - bc = r^2 - (a+d)r + ad - bc = 0.$$

Thus $r = \frac{(a+d)\pm\sqrt{(a+d)^2-4(ad-bc)}}{2} = \frac{p \pm \sqrt{p^2-4q}}{2}$

2 real soln

$p^2-4q = \Delta$

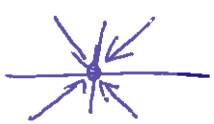
Case 1: $(a+d)^2 - 4(ad-bc) > 0$

$p^2 - 4q > 0$

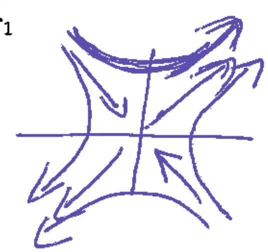
Hence the general solutions is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{r_1 t} + c_2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} e^{r_2 t}$

Case 1a: $r_1 > r_2 > 0$

Case 1b: $r_1 < r_2 < 0$



Case 1c: $r_2 < 0 < r_1$



Δ

Case 2: $(a+d)^2 - 4(ad-bc) = 0$ 1 repeated soln

Case 2i: Two independent eigenvectors:

The general solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{rt} + c_2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} e^{rt}$

Case 2ii: One independent eigenvectors:

The general solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{rt} + c_2 \left[\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} t + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right] e^{rt}$

Case 2a: $r > 0$

Case 2b: $r < 0$

$\Delta = p^2 - 4q$

Case 3: $(a+d)^2 - 4(ad-bc) < 0$. I.e., $r = \lambda \pm i\mu$ 2 complex

Suppose the eigenvector corresponding to this eigenvalue is

$$\begin{pmatrix} v_1 + iw_1 \\ v_2 + iw_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + i \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$\lambda = \frac{a+d}{2}$

Then general solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 cos(\mu t) - w_1 sin(\mu t) \\ v_2 cos(\mu t) - w_2 sin(\mu t) \end{pmatrix} e^{\lambda t} + c_2 \begin{pmatrix} v_1 sin(\mu t) + w_1 cos(\mu t) \\ v_2 sin(\mu t) + w_2 cos(\mu t) \end{pmatrix} e^{\lambda t}$$

Case 3a: $\lambda > 0$

$\mu = \frac{\sqrt{(a+d)^2 - 4(ad-bc)}}{2}$

Case 3a: $\lambda < 0$

Case 3a: $\lambda = 0$