IVP: $\frac{d\vec{x}}{dt} = \vec{x}' = A\vec{x}$, $\vec{x}(t_0) = \begin{bmatrix} c \\ d \end{bmatrix}$

Solve: $\vec{x}' = \begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix}$ has e.vectors $c_1 \begin{bmatrix} -1 \\ 5 \end{bmatrix}$ w/ e.value $= -1$

and e.vectors $c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ w/ e.value $= 5$

Thus general solution is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 5 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t}$$

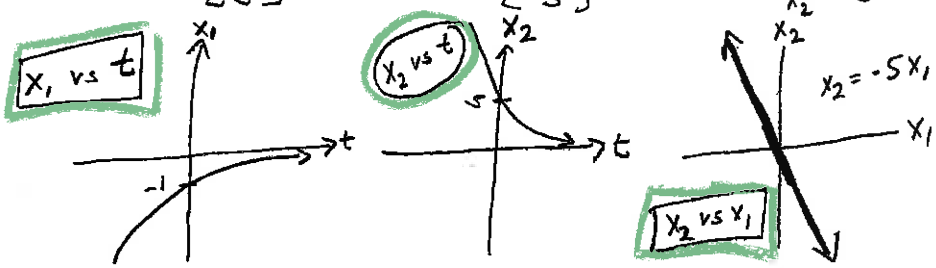
I.V.P.: Suppose $\vec{x}(0) = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$

$$\begin{bmatrix} -1 \\ 5 \end{bmatrix} = \vec{x}(0) = c_1 \begin{bmatrix} -1 \\ 5 \end{bmatrix} e^0 + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^0$$

$0 \neq \begin{vmatrix} -1 & 1 \\ 5 & 1 \end{vmatrix} = W(\phi_1, \phi_2)(0)$ $\Rightarrow$ unique soln to IVP

$$\left.\begin{aligned} -1 &= -c_1 + c_2 \\ 5 &= 5c_1 + c_2 \end{aligned}\right\} \Rightarrow c_1 = 1,\ c_2 = 0$$

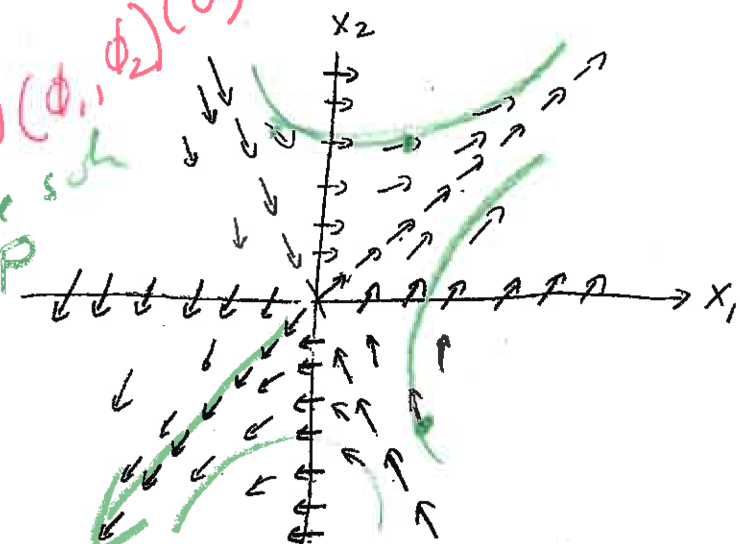
If $\vec{x}(0) = \begin{bmatrix} -1 \\ 5 \end{bmatrix} \Rightarrow \vec{x} = \begin{bmatrix} -1 \\ 5 \end{bmatrix} e^{-t} \Rightarrow \begin{aligned} x_1 &= -e^{-t} \\ x_2 &= 5e^{-t} \end{aligned}$




$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 4x_1 + x_2 \\ 5x_1 \end{bmatrix}$$

$$\frac{dx_2}{dx_1} = \frac{dx_2}{dt} \cdot \frac{dt}{dx_1} = \frac{dx_2/dt}{dx_1/dt} = \frac{x_2'}{x_1'}$$

$= \boxed{\phantom{xx}}$




If $x_2 = -5x_1 \Rightarrow \frac{x_2'}{x_1'} = \frac{5x_1}{4x_1 - 5x_1} = \frac{5x_1}{-x_1} = -5$

Ch 7 and 9

Suppose an object moves in the 2D plane (the $x_1, x_2$ plane) so that it is at the point $(x_1(t), x_2(t))$ at time $t$. Suppose the object's velocity is given by

$$x_1'(t) = ax_1 + bx_2,$$
$$x_2'(t) = cx_1 + dx_2$$

Or in matrix form $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

To solve, find eigenvalues and corresponding eigenvectors:

$|A - rI| = \begin{vmatrix} a-r & b \\ c & d-r \end{vmatrix} = (a-r)(d-r) - bc = r^2 - (a+d)r + ad - bc = 0.$

Thus $r = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$

**Case 1:** $(a+d)^2 - 4(ad-bc) > 0$ 2 real

Hence the general solutions is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{r_1 t} + c_2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} e^{r_2 t}$

Case 1a: $r_1 > r_2 > 0$

$e^{r_i t} \to \infty$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} \pm\infty \\ \pm\infty \end{pmatrix}$

Case 1b: $r_1 < r_2 < 0$

$e^{r_i t} \to 0$

$\Rightarrow \vec{x} = 0$

Case 1c: $r_2 < 0 < r_1$

$e^{r_1 t} \to \pm\infty$

$e^{r_2 t} \to 0$


$\vec{x} = c_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{r_1 t} + c_2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} e^{r_2 t}$ → 0 as $t \to \infty$

$\begin{vmatrix} v_1 e^{rt} & w_1 e^{rt} \\ v_2 e^{rt} & w_2 e^{rt} \end{vmatrix} = e^{2rt}(v_1 w_2 - v_2 w_1) \neq 0$

assuming $\vec{v}, \vec{w}$ are l.i.

one repeated root

**Case 2:** $(a+d)^2 - 4(ad-bc) = 0$

Case 2i: Two independent eigenvectors:

The general solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{rt} + c_2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} e^{rt}$ ← 2-dim e. space

Case 2ii: One independent eigenvectors:

The general solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{rt} + c_2 \left[\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} t + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right] e^{rt}$

Case 2a: $r > 0$

Case 2b: $r < 0$

**Case 3:** $(a+d)^2 - 4(ad-bc) < 0$. I.e., $r = \lambda \pm i\mu$ ← 2 complex sol'ns

Suppose the eigenvector corresponding to this eigenvalue is

$\begin{pmatrix} v_1 + iw_1 \\ v_2 + iw_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + i \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

Then general solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 cos(\mu t) - w_1 sin(\mu t) \\ v_2 cos(\mu t) - w_2 sin(\mu t) \end{pmatrix} e^{\lambda t} + c_2 \begin{pmatrix} v_1 sin(\mu t) + w_1 cos(\mu t) \\ v_2 sin(\mu t) + w_2 cos(\mu t) \end{pmatrix} e^{\lambda t}$$

Case 3a: $\lambda > 0$

Case 3a: $\lambda < 0$

Case 3a: $\lambda = 0$

Derivation of general solutions:

---

If $b^2 - 4ac > 0$ we guessed $e^{rt}$ is a solution and noted that any linear combination of solutions is a solution to a homogeneous linear differential equation.

---

Section 3.3: If $b^2 - 4ac < 0$, :

Changed format of $y = c_1e^{r_1t} + c_2e^{r_2t}$ to linear combination of real-valued functions instead of complex valued functions by using Euler's formula:

$$e^{it} = cos(t) + isin(t)$$

Hence $e^{(d+in)t} = e^{dt}e^{int} = e^{dt}[cos(nt) + isin(nt)]$

Let $r_1 = d + in$, $r_2 = d - in$

$$y = c_1e^{r_1t} + c_2e^{r_2t}$$

$$= c_1e^{dt}[cos(nt)+isin(nt)]+c_2e^{dt}[cos(-nt)+isin(-nt)]$$

$$= c_1e^{dt}cos(nt)+ic_1e^{dt}sin(nt)+c_2e^{dt}cos(nt)-ic_2e^{dt}sin(nt)$$

$$=(c_1 + c_2)e^{dt}cos(nt) + i(c_1 - c_2)e^{dt}sin(nt)$$

$$= k_1e^{dt}cos(nt) + k_2e^{dt}sin(nt)$$

---



Section 3.4: If $b^2 - 4ac = 0$, then $r_1 = r_2$.
Hence one solution is $y = e^{r_1t}$ Need second solution.

If $y = e^{rt}$ is a solution, $y = ce^{rt}$ is a solution.

How about $y = v(t)e^{rt}$?

$$y' = v'(t)e^{rt} + v(t)re^{rt}$$

$$y'' = v''(t)e^{rt} + v'(t)re^{rt} + v'(t)re^{rt} + v(t)r^2e^{rt}$$

$$= v''(t)e^{rt} + 2v'(t)re^{rt} + v(t)r^2e^{rt}$$

$$ay'' + by' + cy = 0$$

$$a(v''e^{rt}+2v're^{rt}+vr^2e^{rt})+b(v'e^{rt}+vre^{rt})+cve^{rt} = 0$$

$$a(v''(t)+2v'(t)r+v(t)r^2)+b(v'(t)+v(t)r)+cv(t) = 0$$

$$av''(t)+2av'(t)r+av(t)r^2+bv'(t)+bv(t)r+cv(t) = 0$$

$$av''(t) + (2ar + b)v'(t) + (ar^2 + br + c)v(t) = 0$$

$$av''(t) + (2a(\tfrac{-b}{2a}) + b)v'(t) + 0 = 0$$

since $ar^2 + br + c = 0$ and $r = \frac{-b}{2a}$

$av''(t) + (-b + b)v'(t) = 0$. Thus $av''(t) = 0$.

Hence $v''(t) = 0$ and $v'(t) = k_1$ and $v(t) = k_1t + k_2$

Hence $v(t)e^{r_1t} = (k_1t + k_2)e^{r_1t}$ is a soln

Thus $te^{r_1t}$ is a nice second solution.

Hence general solution is $y = c_1e^{r_1t} + c_2te^{r_1t}$

$mr'' = \frac{-GMm}{r^2}$

Let $v = r'$, then $v' = r''$

Thus we obtain system of non-linear equations:

$$r' = v$$
$$v' = \frac{-GM}{r^2}$$

Note $v' = \frac{-GM}{r^2}$ involves 3 variables: $v, t, r$

Eliminate $t$: $v' = \frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = \frac{dv}{dr}v$

Thus $mv' = \frac{-GMm}{r^2}$ becomes $m\frac{dv}{dr}v = \frac{-GMm}{r^2}$

Separate variables: $\int mdvv = \int \frac{-GMm}{r^2}dr$

$$\frac{1}{2}mv^2 = \frac{GMm}{r} + E \text{ where } E \text{ is a constant.}$$

Thus we have derived the physics formula, conservation of energy:

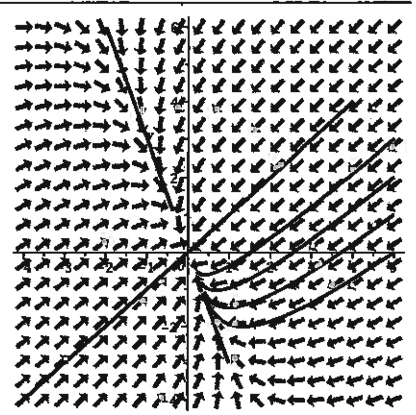
$$\frac{1}{2}mv^2 + \frac{-GMm}{r} = E$$

I.e., Kinetic Energy + Potential Energy = constant

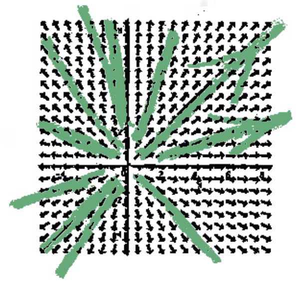
$m = -\frac{1}{3}$

---

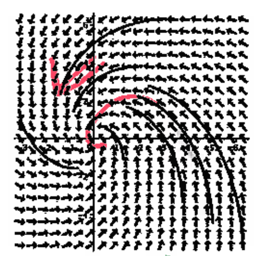
$x' = -4x - y$
$y' = -3x + 2y$



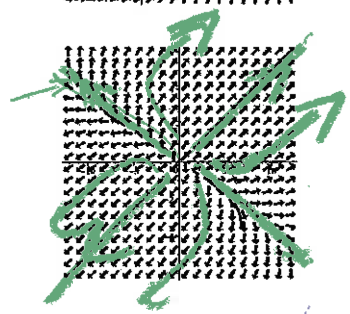
Suppose the following represent direction fields of linear systems of 1rst order differential equations in the phase plane. What can you say about solutions to these systems of equations.



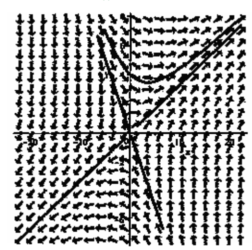
~~Asym~~ unstable
1 repeated positive real e. value



asym stable
complex e. value $a+bi$
$a < 0$



2 real positive e. value
$r_1$ w/ e. vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$r_2$ w/ e. vector $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$
unstable



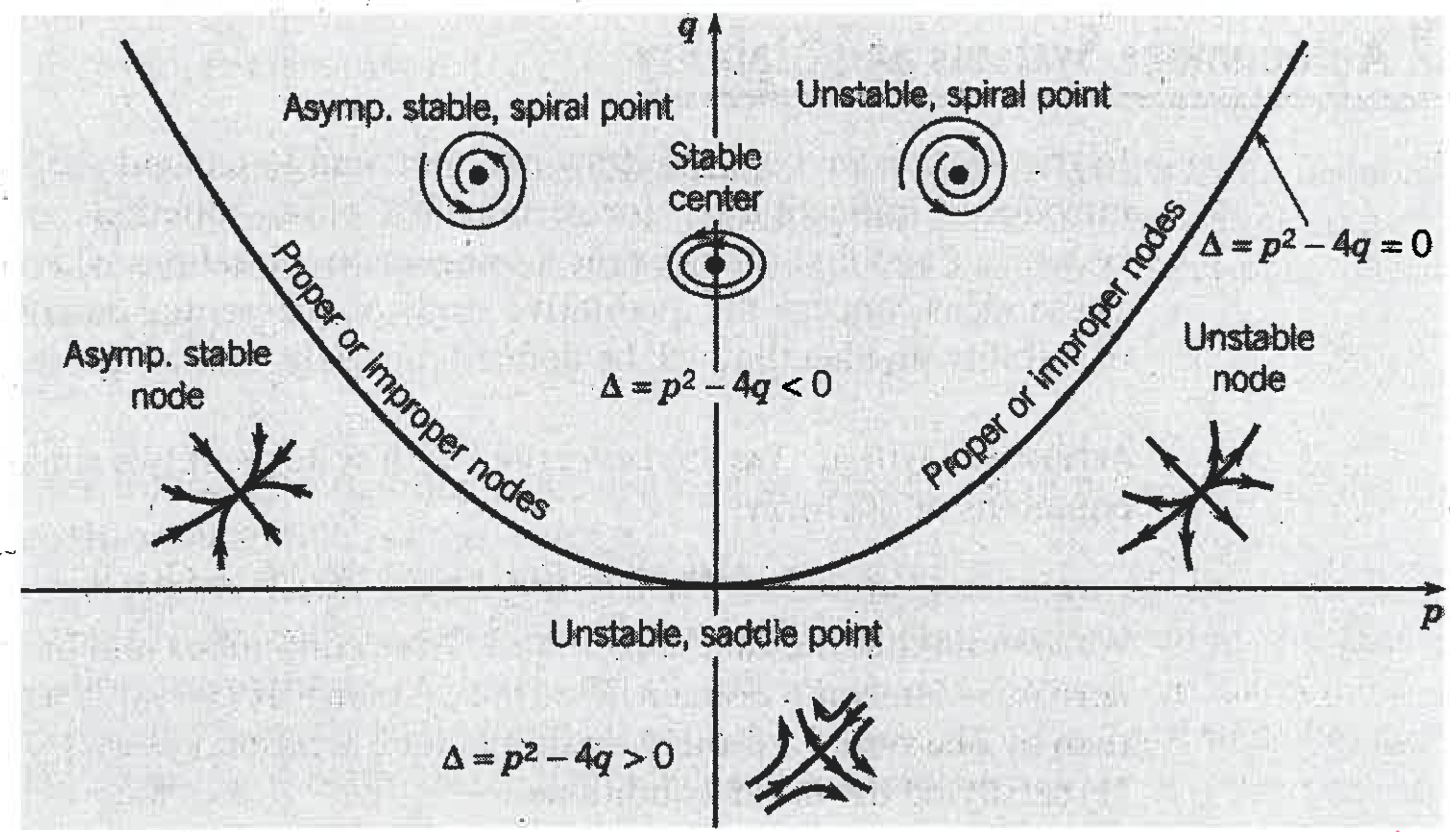
unstable

(a) Node if $q > 0$ and $\Delta \geq 0$; (b) Saddle point if $q < 0$;
(c) Spiral point if $p \neq 0$ and $\Delta < 0$; (d) Center if $p = 0$ and $q > 0$.

*Hint:* These conclusions can be reached by studying the eigenvalues $r_1$ and $r_2$. It may also be helpful to establish, and then to use, the relations $r_1 r_2 = q$ and $r_1 + r_2 = p$.

21. Continuing Problem 20, show that the critical point $(0, 0)$ is
(a) Asymptotically stable if $q > 0$ and $p < 0$;
(b) Stable if $q > 0$ and $p = 0$;
(c) Unstable if $q < 0$ or $p > 0$.

The results of Problems 20 and 21 are summarized visually in Figure 9.1.9.




$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

**FIGURE 9.1.9** Stability diagram.

$\boxed{\Delta = (a+d)^2 - 4(ad-bc)}$

$p = a + d$

$q = ad - bc$