Section 6.3

Example: $f(t) = \begin{cases} f_1, & \text{if } t < 4; \\ f_2, & \text{if } 4 \le t < 5; \\ f_3, & \text{if } 5 \le t < 10; \\ f_4, & \text{if } t \ge 10; \end{cases}$

Hence $f(t) = f_1(t) + u_4(t)[f_2(t) - f_1(t)] + u_5(t)[f_3(t) - f_2(t)] + u_{10}(t)[f_4(t) - f_3(t)]$

Formula 13: $\mathcal{L}(u_c(t)f(t-c)) = e^{-cs}\mathcal{L}(f(t)). = e^{-cs}F(s)$

or equivalently

$$\mathcal{L}(u_c(t)f(t-c+c)) = e^{-cs}\mathcal{L}(f(t+c)).$$

or equivalently

$$\mathcal{L}(u_c(t)f(t)) = e^{-cs}\mathcal{L}(f(t+c)).$$

In other words, replacing $t-c$ with t is equivalent to replacing t with $t+c$

Formula 13: $\mathcal{L}^{-1}(\mathcal{L}(u_c(t)f(t-c))) = \mathcal{L}^{-1}(e^{-cs}\mathcal{L}(f(t))). = \mathcal{L}^{-1}(e^{-cs}F(s))$

Let $F(s) = \mathcal{L}(f(t))$. Then $\mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}(\mathcal{L}(f(t))) = f(t)$.

Thus $\mathcal{L}^{-1}(e^{-cs}F(s)) = \mathcal{L}^{-1}(e^{-cs}\mathcal{L}(f(t))) = u_c(t)f(t-c)$ where $f(t) = \mathcal{L}^{-1}(F(s))$ ∎

$$F(s) = \mathcal{L}(f(t))$$

$$\mathcal{L}^{-1}(F(s)) = f(t)$$

6.6: The Convolution Integral

Defn: The convolution of f and g is the function $f * g$ defined by

$$\boxed{(f * g)(t) = \int_0^t f(t-s)g(s)ds = \int_0^t f(x)g(t-x)dx}$$

Note $*$ is

1.) commutative: $f * g = g * f$

2.) associative: $(f * g) * h = f * (g * h)$

3.) distributive w.r.t +: $f * (g_1 + g_2) = f * g_1 + f * g_2$

4.) $f * 0 = 0 * f = 0$.

Example: $cos(t) * 1 =$ $\sin t$

Example: $t * t \ngeq 0$

$$t * t = \int_0^t (t-x)\, x \, dx$$

$$= \int_0^t (\cancel{t}x - x^2)dx$$

$$= \frac{tx^2}{2} - \frac{x^3}{3}\Big|_0^t = \frac{t^3}{2} - \frac{t^3}{3} - 0 = \frac{3t^3 - 2t^3}{6} = \frac{t^3}{6}$$

Thm: $\mathcal{L}((f * g)(t)) = \mathcal{L}(f(t)) \cdot \mathcal{L}(g(t))$

Proof:

$\mathcal{L}(f(t))\mathcal{L}(g(t)) = \int_0^\infty e^{-sy} f(y)dy \int_0^\infty e^{-sx} g(x)dx$

$= \int_0^\infty [\int_0^\infty e^{-sy} f(y)dy] e^{-sx} g(x)dx$

$= \int_0^\infty [\int_0^\infty e^{-sy} f(y) e^{-sx} g(x)dy]dx$

$= \int_0^\infty [\int_0^\infty e^{-s(y+x)} f(y)g(x)dy]dx$

$= \int_0^\infty [\int_0^\infty e^{-s(y+x)} f(y)g(x)dx]dy$

Let $t = x + y$, $dt = dx$

$= \int_0^\infty [\int_y^\infty e^{-st} f(y)g(t-y)dt]dy$

$= \int_0^\infty [\int_0^t e^{-st} f(y)g(t-y)dy]dt$

$= \int_0^\infty e^{-st} [\int_0^t f(y)g(t-y)dy]dt$

$= \int_0^\infty e^{-st} (f * g)(t)dt$

$= \mathcal{L}(f * g)$

Example: $\mathcal{L}^{-1}(\frac{1}{s(s-a)}) =$