Let $\delta = \lim_{k \to 0} d_k$

6.5: Impulse functions

Unit impulse function = Dirac delta function is a generalized function with the properties

$$\boxed{\delta(t) = 0, \quad t \neq 0 \qquad \int_{-\infty}^{\infty} \delta(t)dt = 1 \qquad \mathcal{L}(\delta(t-t_0)) = e^{-st_0}}$$

Let $d_k(t) = \begin{cases} \frac{1}{2k} & -k < t < k \\ 0 & t \leq -k \text{ or } t \geq k \end{cases}$

Note $lim_{k \to 0} d_k(t) = 0$ if $t \neq 0$

and $lim_{k \to 0} \int_{-\infty}^{\infty} d_k(t) = lim_{k \to 0} 1 = 1 = \int_{-\infty}^{\infty} \delta(t)dt$

$\mathcal{L}(\delta(t-t_0)) = lim_{k \to 0} \mathcal{L}(d_k(t-t_0))$

$= lim_{k \to 0} \int_0^{\infty} e^{-st} d_k(t-t_0)dt$

$= lim_{k \to 0} \frac{1}{2k} \int_{t_0-k}^{t_0+k} e^{-st}dt$

$= lim_{k \to 0} \frac{-1}{2sk} e^{-st}|_{t_0-k}^{t_0+k}$

$= lim_{k \to 0} \frac{1}{2sk} e^{-st_0}(e^{sk} - e^{-sk})$

$= lim_{k \to 0} \frac{sinh(sk)}{sk} e^{-st_0}$

$= lim_{k \to 0} \frac{scosh(sk)}{s} e^{-st_0} = e^{-st_0}$

$\mathcal{L}(\delta(t-t_0)) = e^{-st_0}$

$sin(t) = \frac{e^{it}-e^{-it}}{2i}$ $\qquad cos(t) = \frac{e^{it}+e^{-it}}{2}$

$sinh(t) = \frac{e^{t}-e^{-t}}{2}$ $\qquad cosh(t) = \frac{e^{t}+e^{-t}}{2}$

$[sinh(t)]' =$ $\qquad [cosh(t)]' =$

$sinh(0) = \frac{e^0-e^0}{2} = 0$ $\qquad cosh(0) = \frac{e^0+e^0}{2} = 1$

Intro to Group Theory

Define the $\cdot$ product on R^2 by

$(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 - y_1x_2)$

Note $\cdot$ is

1.) commutative:
$(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 - y_1x_2)$
$= (x_2x_1 - y_2y_1, x_2y_1 - y_2x_1) = (x_2, y_2) \cdot (x_1, y_1)$

2.) associative: $(f \cdot g) \cdot h = f \cdot (g \cdot h)$

3.) distributive w.r.t +: $f \cdot (g_1 + g_2) = f \cdot g_1 + f \cdot g_2$

4.) $(x_1, y_1) \cdot (0, 0) = (0, 0)$

Note $(0, 1) \cdot (0, 1) = (-1, 0)$

Section 6.3

Example: $f(t) = \begin{cases} f_1, & \text{if } t < 4; \\ f_2, & \text{if } 4 \leq t < 5; \\ f_3, & \text{if } 5 \leq t < 10; \\ f_4, & \text{if } t \geq 10; \end{cases}$

Hence $f(t) = f_1(t) + u_4(t)[f_2(t) - f_1(t)] + u_5(t)[f_3(t) - f_2(t)] + u_{10}(t)[f_4(t) - f_3(t)]$

Formula 13: $\mathcal{L}(u_c(t)f(t-c)) = e^{-cs}\mathcal{L}(f(t)).$ $= e^{-cs}F(s)$

or equivalently

$$\mathcal{L}(u_c(t)f(t-c+c)) = e^{-cs}\mathcal{L}(f(t+c)).$$

or equivalently

$$\mathcal{L}(u_c(t)f(t)) = e^{-cs}\mathcal{L}(f(t+c)).$$

In other words, replacing $t-c$ with t is equivalent to replacing t with $t+c$

Formula 13: $\mathcal{L}^{-1}(\mathcal{L}(u_c(t)f(t-c))) = \mathcal{L}^{-1}(e^{-cs}\mathcal{L}(f(t))) = \mathcal{L}^{-1}(e^{-cs}F(s))$

Let $F(s) = \mathcal{L}(f(t))$. Then $\mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}(\mathcal{L}(f(t))) = f(t)$.

Thus $\mathcal{L}^{-1}(e^{-cs}F(s)) = \mathcal{L}^{-1}(e^{-cs}\mathcal{L}(f(t))) = u_c(t)f(t-c)$ where $f(t) = \mathcal{L}^{-1}(F(s))$ ∎

$F(s) = \mathcal{L}(f(t))$

$\mathcal{L}^{-1}(F(s)) = f(t)$