homogeneous <u>LINEAR</u> eqns

Nullspace is a vector space

Defn: A set V together with two operations, called addition and scalar multiplication is a **vector space** if the following vector space axioms are satisfied for all vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ in V and all scalars, c, d in R.

Vector space axioms:

a.) $\mathbf{u} + \mathbf{v}$ is in V

b.) $c\mathbf{u}$ is in V

c.) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

d.) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

e.) There is a vector, denoted by $\mathbf{0}$, in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u}$ in V

f.) For each $\mathbf{u}$ in V, there is an element, denoted by $-\mathbf{u}$, in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

g.) $(cd)\mathbf{u} = c(d\mathbf{u})$

h.) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

i.) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

j.) $1\mathbf{u} = \mathbf{u}$

Examples:

1.) R^k with the usual operations of addition and scalar multiplication is a vector space.

2.) The set $M^{k,n}$, the set of all $k \times n$ matrices with the usual operations of addition and scalar multiplication is a vector

Linear Algebra Review: Eigenvalues and Eigenvectors

Defn: λ is an **eigenvalue** of the linear transformation $T : V \to V$ if there exists a <u>nonzero</u> vector $\mathbf{x}$ in V such that $T(\mathbf{x}) = \lambda\mathbf{x}$. The vector $\mathbf{x}$ is said to be an **eigenvector** corresponding to the eigenvalue λ.

e. vector $\vec{x} \neq \vec{0}$

Example: Let $T(\mathbf{x}) = \begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \mathbf{x}$.

$A\vec{x} = \lambda \vec{x}$

e. value

Note $\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix} = -1 \begin{bmatrix} -1 \\ 5 \end{bmatrix}$

Thus -1 is an eigenvalue of A and $\begin{bmatrix} -1 \\ 5 \end{bmatrix}$ is a corresponding eigenvector of A.

Note $\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Thus 5 is an eigenvalue of A and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a corresponding eigenvector of A.

Note $\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \end{bmatrix} \neq k \begin{bmatrix} 2 \\ 8 \end{bmatrix}$ for any k.

Thus $\begin{bmatrix} 2 \\ 8 \end{bmatrix}$ is <u>NOT</u> an eigenvector of A.

MOTIVATION:

Note $\begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Thus $A\begin{bmatrix} 2 \\ 8 \end{bmatrix} = A(\begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = A\begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3A\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$= -1\begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3 \cdot 5\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \end{bmatrix}$

Finding eigenvalues:

Suppose $A\mathbf{x} = \lambda\mathbf{x}$ (Note A is a SQUARE matrix).

Then $A\mathbf{x} = \lambda I\mathbf{x}$ where I is the identity matrix. $\cancel{\lambda I x - A x = 0}$

Thus $\lambda I\mathbf{x} - A\mathbf{x} = (\lambda I - A)\mathbf{x} = \mathbf{0}$

Thus if $A\mathbf{x} = \lambda\mathbf{x}$ for a nonzero $\mathbf{x}$, then $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has a nonzero solution.

Thus $det(\lambda I - A)\mathbf{x} = 0$.

Note that the eigenvectors corresponding to λ are the nonzero solutions of $(\lambda I - A)\mathbf{x} = \mathbf{0}$.

$Ax - \lambda I x = 0$

$(A - \lambda I)\vec{x} = 0$

Solve $|A - \lambda I| = 0$ to find e. values

Thus to find the eigenvalues of A and their corresponding eigenvectors:

Step 1: Find eigenvalues: Solve the equation

$$det(\lambda I - A) = 0 \text{ for } \lambda.$$

Step 2: For each eigenvalue λ_0, find its corresponding eigenvectors by solving the homogeneous system of equations

$$(\lambda_0 I - A)\mathbf{x} = 0 \text{ for } \mathbf{x}.$$

Defn: $det(\lambda I - A) = 0$ is the **characteristic equation** of A.

Thm 3: The eigenvalues of an upper triangular or lower triangular matrix (including diagonal matrices) are identical to its diagonal entries.

Defn: The **eigenspace** corresponding to an eigenvalue λ_0 of a matrix A is the set of all solutions of $(\lambda_0 I - A)\mathbf{x} = \mathbf{0}$.

Note: An eigenspace is a vector space

The vector $\mathbf{0}$ is always in the eigenspace.

The vector $\mathbf{0}$ is never an eigenvector.

The number 0 can be an eigenvalue.

Thm: A square matrix is invertible if and only if $\lambda = 0$ is not an eigenvalue of A.

6.2

R

The LaPlace Transform is a method to change a differential equation to a linear equation.

Example: Solve $y'' + 3y' + 4y = 0,\ y(0) = 5,\ y'(0) = 6$

1.) Take the LaPlace Transform of both sides of the equation:

$\mathcal{L}(y'' + 3y' + 4y) = \mathcal{L}(0)$

2.) Use the fact that the LaPlace Transform is linear:

$\mathcal{L}(y'') + 3\mathcal{L}(y') + 4\mathcal{L}(y) = 0$

3.) Use thm to change this equation into an algebraic equation:

$[s^2\mathcal{L}(y) - sy(0) - y'(0)] + 3[s\mathcal{L}(y) - y(0)] + 4\mathcal{L}(y) = 0$

3.5) Substitute in the initial values:

$s^2\mathcal{L}(y) - 5s - 6 + 3[s\mathcal{L}(y) - 5] + 4\mathcal{L}(y) = 0$

4.) Solve the algebraic equation for $\mathcal{L}(y)$

$s^2\mathcal{L}(y) - 5s - 6 + 3s\mathcal{L}(y) - 15 + 4\mathcal{L}(y) = 0$

$[s^2 + 3s + 4]\mathcal{L}(y) = 5s + 21$

$\mathcal{L}(y) = \frac{5s+21}{s^2+3s+4}$

Some algebra implies $\mathcal{L}(y) = \frac{5s+21}{s^2+3s+4}$

5.) Solve for y by taking the inverse LaPlace transform of both sides (use a table):

$\mathcal{L}^{-1}(\mathcal{L}(y)) = \mathcal{L}^{-1}(\frac{5s+21}{s^2+3s+4})$

$y = \mathcal{L}^{-1}(\frac{5s+21}{s^2+3s+4})$

Find the inverse LaPlace transform of $\frac{5s+21}{s^2+3s+4}$

Look at the denominator first to determine if it is of the form $s^2 \pm a^2$ or $(s-a)^{n+1}$ or $(s-a)^2 + b^2$ OR if you should factor and use partial fractions

$s^2 + 3s + 4$: $b^2 - 4ac = 3^2 - 4(1)(4) = 9 - 16 < 0$

Hence s^2+3s+4 does not factor over the reals. Hence to avoid complex numbers, we won't factor it.

s^2+3s+4 is not an s^2-a^2 or an s^2+a^2 or an $(s-a)^2$, so it must be an $(s-a)^2 + b^2$.

Hence we will complete the square:

$s^2 + 3s + \_\_\_ - \_\_\_ + 4 = (s + \_\_\_)^2 - \_\_\_ + 4$

Hence $\frac{5s+21}{s^2+3s+4} = \frac{5s+21}{(s+\frac{3}{2})^2+\frac{7}{4}}$

Must now consider the numerator. We need it to look like $s - a = s + \frac{3}{2}$ or $b = \sqrt{\frac{7}{4}}$ in order to use $\mathcal{L}^{-1}(\frac{s-a}{(s-a)^2+b^2}) = e^{at}cosbt$
and/or $\mathcal{L}^{-1}(\frac{b}{(s-a)^2+b^2}) = e^{at}sinbt$

$5s + 21 = 5(s + \frac{3}{2}) - \frac{15}{2} + 21 = 5(s + \frac{3}{2}) - \frac{27}{2}$

$= 5(s + \frac{3}{2}) - [\frac{27}{2}\sqrt{\frac{4}{7}}]\sqrt{\frac{7}{4}} = 5(s + \frac{3}{2}) - [\frac{27}{\sqrt{7}}]\sqrt{\frac{7}{4}}$

Hence $\frac{5s+21}{s^2+3s+4} = \frac{5(s+\frac{3}{2})-[\frac{27}{\sqrt{7}}]\sqrt{\frac{7}{4}}}{(s+\frac{3}{2})^2+\frac{7}{4}}$

$= 5[\frac{s+\frac{3}{2}}{(s+\frac{3}{2})^2+\frac{7}{4}}] - \frac{27}{\sqrt{7}}[\frac{\sqrt{\frac{7}{4}}}{(s+\frac{3}{2})^2+\frac{7}{4}}]$

Thus $\mathcal{L}^{-1}(\frac{5s+21}{s^2+3s+4}) = \mathcal{L}^{-1}(5[\frac{s+\frac{3}{2}}{(s+\frac{3}{2})^2+\frac{7}{4}}] - \frac{27}{\sqrt{7}}[\frac{\sqrt{\frac{7}{4}}}{(s+\frac{3}{2})^2+\frac{7}{4}}])$

$= 5\mathcal{L}^{-1}(\frac{s+\frac{3}{2}}{(s+\frac{3}{2})^2+\frac{7}{4}}) - \frac{27}{\sqrt{7}}\mathcal{L}^{-1}(\frac{\sqrt{\frac{7}{4}}}{(s+\frac{3}{2})^2+\frac{7}{4}})$

$= 5e^{-\frac{3}{2}t}cos\sqrt{\frac{7}{4}}t - \frac{27}{\sqrt{7}}e^{-\frac{3}{2}t}sin\sqrt{\frac{7}{4}}t$

Hence $y(t) = 5e^{-\frac{3}{2}t}cos\sqrt{\frac{7}{4}}t - \frac{27}{\sqrt{7}}e^{-\frac{3}{2}t}sin\sqrt{\frac{7}{4}}t.$

TABLE 6.2.1 Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	Notes
1. 1	$\frac{1}{s}, \quad s > 0$	Sec. 6.1; Ex. 4
2. e^{at} $\mathcal{L}(t) = \frac{1}{s^2}$	$\frac{1}{s-a}, \quad s > a$	Sec. 6.1; Ex. 5
3. t^n, $n =$ positive integer	$\frac{n!}{s^{n+1}}, \quad s > 0$	Sec. 6.1; Prob. 27
4. t^p, $p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$	Sec. 6.1; Prob. 27
5. $\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Ex. 6
6. $\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Prob. 6
7. $\sinh at$	$\frac{a}{s^2-a^2}, \quad s > \lvert a \rvert$	Sec. 6.1; Prob. 8
8. $\cosh at$	$\frac{s}{s^2-a^2}, \quad s > \lvert a \rvert$	Sec. 6.1; Prob. 7
9. $e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 13
10. $e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 14
11. $t^n e^{at}$, $n =$ positive integer	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$	Sec. 6.1; Prob. 18
12. $u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$	Sec. 6.3
13. $u_c(t)f(t-c)$	$e^{-cs}F(s) = e^{-cs}\mathcal{L}(f(t))$	Sec. 6.3
14. $e^{ct}f(t)$	$F(s-c)$	Sec. 6.3
15. $f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right), \quad c > 0$	Sec. 6.3; Prob. 19
16. $\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	Sec. 6.6
17. $\delta(t-c)$	e^{-cs}	Sec. 6.5
18. $f^{(n)}(t)$	$s^n F(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	Sec. 6.2
19. $(-t)^n f(t)$	$F^{(n)}(s)$	Sec. 6.2; Prob. 28

$\mathcal{L}(y'') = s^2\mathcal{L}(y) - s\,y(0) - 1\,y'(0)$

6.4

$$g(t) = \begin{cases} 0 & t < 4 \\ 2 & 4 \le t < 10 \\ t & t \ge 10 \end{cases}$$

Hence $g(t) = 2u_4(t) + (t-2)u_{10}(t)$

$$\boxed{\text{Solve } 3y'' + y' + y = 2u_4(t) + (t-2)u_{10}(t),\quad y(0) = 0,\ y'(0) = 0}$$

take Laplace transform of both sides

$3\mathcal{L}(y'') + \mathcal{L}(y') + \mathcal{L}(y) = \mathcal{L}(2u_4(t)) + \mathcal{L}((t-2)u_{10}(t))$

Thm: $\mathcal{L}(u_c(t)f(t-c)) = e^{-cs}\mathcal{L}(f(t))$.

Thus $\mathcal{L}(u_c(t)f(t)) = e^{-cs}\mathcal{L}(f(t+c))$

$3[s^2\mathcal{L}(y) - sy(0) - y'(0)] + s\mathcal{L}(y) - y(0) + \mathcal{L}(y)$
$= e^{-4s}\mathcal{L}(2) + e^{-10s}\mathcal{L}((t+8))$

$3[s^2\mathcal{L}(y)] + s\mathcal{L}(y) + \mathcal{L}(y) = 2e^{-4s}\mathcal{L}(1) + e^{-10s}\mathcal{L}(t) + 8e^{-10s}\mathcal{L}(1)$

$\mathcal{L}(y)[3s^2 + s + 1] = e^{-4s}\frac{2}{s} + e^{-10s}\frac{1}{s^2} + e^{-10s}\frac{8}{s}$

$\mathcal{L}(y) = e^{-4s}\frac{2}{s[3s^2+s+1]} + e^{-10s}\frac{1}{s^2[3s^2+s+1]} + 8e^{-10s}\frac{1}{s[3s^2+s+1]}$

$y = 2\mathcal{L}^{-1}(e^{-4s}\frac{1}{s[3s^2+s+1]}) + \mathcal{L}^{-1}(e^{-10s}\frac{1}{s^2[3s^2+s+1]})$
$+8\mathcal{L}^{-1}(e^{-10s}\frac{1}{s[3s^2+s+1]})$

solve for y

Note this is our characteristic eqn

$y = u_4(t)f(t-4) + u_{10}h(t-10) + 8u_{10}f(t-10)$

where $f(t) = \mathcal{L}^{-1}(\frac{1}{s[3s^2+s+1]})$ and $h(t) = \mathcal{L}^{-1}(\frac{1}{s^2[3s^2+s+1]})$

$\frac{1}{s[3s^2+s+1]} = \frac{A}{s} + \frac{Bs+C}{3s^2+s+2}$

$1 = A(3s^2+s+1) + (Bs+C)s$

$0s^2 + 0s + 1 = (3A+B)s^2 + (A+C)s + A$

$0 = 3A + B,\ 0 = A + C,\ 1 = A$

Hence $A = 1,\ B = -3A = -3,\ C = -A = -1$

$f(t) = \mathcal{L}^{-1}(\frac{1}{s[3s^2+s+1]})$

$= \mathcal{L}^{-1}(\frac{1}{s} + \frac{-3s-1}{3s^2+s+1})$

$= \mathcal{L}^{-1}(\frac{1}{s} + \frac{-3s-1}{3s^2+s+1})$

$= 1 + \mathcal{L}^{-1}(\frac{-3s-1}{3[s^2+\frac{1}{3}s+\frac{1}{3}]}))$

$= 1 + \mathcal{L}^{-1}(\frac{-3s-1}{3[(s^2+\frac{1}{3}s+\_\_\_\_)-\_\_\_\_+\frac{1}{3}]})$

$= 1 + \mathcal{L}^{-1}(\frac{-3s-1}{3[(s+\frac{1}{6})^2-\frac{1}{36}+\frac{1}{3}]})$

$= 1 + \mathcal{L}^{-1}(\frac{-3(s+\frac{1}{3})}{3[(s+\frac{1}{6})^2+\frac{11}{36}]})$

$= 1 + \mathcal{L}^{-1}(\frac{-(s+\frac{1}{6}-\frac{1}{6}+\frac{1}{3})}{[(s+\frac{1}{6})^2+\frac{11}{36}]})$