By finding e. values & e. vectors

Solve: $\vec{x}\,' = \begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\left.\begin{array}{l} \begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \text{ has e. vectors } c_1 \begin{bmatrix} -1 \\ 5 \end{bmatrix} \text{ w/ e. value} = -1 \\ \text{and e. vectors } c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ w/ e. value} = 5 \end{array}\right\}$

$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 4x_1 + x_2 \\ 5x_1 \end{bmatrix}$

$$\frac{dx_2}{dx_1} = \frac{dx_2}{dt} \cdot \frac{dt}{dx_1} = \frac{dx_2/dt}{dx_1/dt} = \frac{x_2'}{x_1'}$$

$$= \boxed{}$$

Thus general solution is

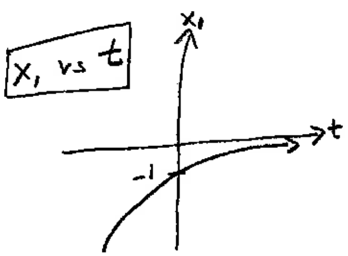
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 5 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t}$$

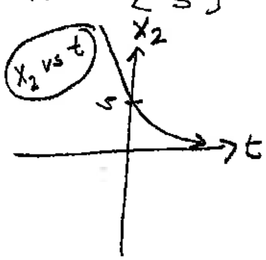
I. V. P.: Suppose $\vec{x}(0) = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$

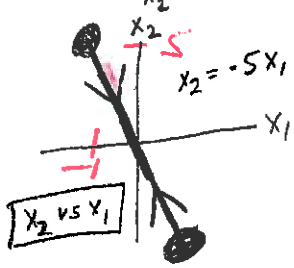
$\begin{bmatrix} -1 \\ 5 \end{bmatrix} = \vec{x}(0) = c_1 \begin{bmatrix} -1 \\ 5 \end{bmatrix} e^0 + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^0$

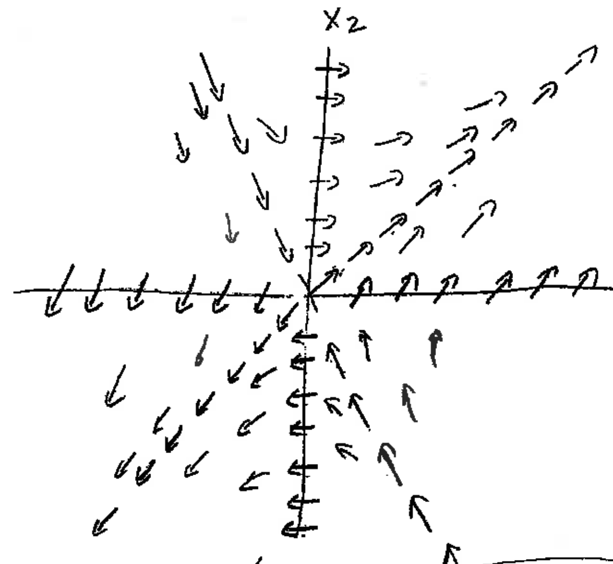
$\left.\begin{array}{l} -1 = -c_1 + c_2 \\ 5 = 5c_1 + c_2 \end{array}\right\} \Rightarrow c_1 = 1, \ c_2 = 0$

If $\vec{x}(0) = \begin{bmatrix} -1 \\ 5 \end{bmatrix} \Rightarrow \vec{x} = \begin{bmatrix} -1 \\ 5 \end{bmatrix} e^{-t} \Rightarrow \begin{array}{l} x_1 = -e^{-t} \\ x_2 = 5e^{-t} \end{array}$









$x_1 = e^{5t}$

$x_2 = e^{5t}$

$x_2 = 1x_1 = \frac{1}{1}x_2$

If $x_2 = -5x_1 \Rightarrow \frac{x_2'}{x_1'} = \frac{5x_1}{4x_1 - 5x_1} = \frac{5x_1}{-x_1} = -5$

Ch 7 and 9

Suppose an object moves in the 2D plane (the x_1, x_2 plane) so that it is at the point $(x_1(t), x_2(t))$ at time t. Suppose the object's velocity is given by

$$x_1'(t) = ax_1 + bx_2,$$
$$x_2'(t) = cx_1 + dx_2$$

Or in matrix form $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

To solve, find eigenvalues and corresponding eigenvectors:

$$|A-rI| \quad \begin{vmatrix} a-r & b \\ c & d-r \end{vmatrix} = (a-r)(d-r) - bc = r^2 - (a+d)r + ad - bc = 0.$$

Thus $r = \frac{(a+d)\pm\sqrt{(a+d)^2-4(ad-bc)}}{2}$

Case 1: $(a+d)^2 - 4(ad-bc) > 0$

Hence the general solutions is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}e^{r_1 t} + c_2\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}e^{r_2 t}$

Case 1a: $r_1 > r_2 > 0$

$e^{r_1 t} \to \infty$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} \pm\infty \\ \pm\infty \end{pmatrix}$

Case 1b: $r_1 < r_2 < 0$

$e^{r_1 t} \to 0$

$\Rightarrow \vec{x} = 0$

Case 1c: $r_2 < 0 < r_1$

$e^{r_1 t} \to \pm\infty$

$e^{r_2 t} \to 0$

$r_2 < 0$

$+$ e. value

$r_1 > 0$

$\vec{x} = c_1\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}e^{r_1 t} + c_2\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}e^{r_2 t}$ 0 as $t \to \infty$

Case 2: $(a+d)^2 - 4(ad-bc) = 0$ one repeated root

Case 2i: Two independent eigenvectors:

The general solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}e^{rt} + c_2\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}e^{rt}$

Case 2ii: One independent eigenvectors:

The general solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}e^{rt} + c_2\left[\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}t + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right]e^{rt}$

Case 2a: $r > 0$

Case 2b: $r < 0$

Case 3: $(a+d)^2 - 4(ad-bc) < 0$. I.e., $r = \lambda \pm i\mu$ $\leftarrow$ 2 complex sol'ns

Suppose the eigenvector corresponding to this eigenvalue is

$$\begin{pmatrix} v_1 + iw_1 \\ v_2 + iw_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + i\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

Then general solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1\begin{pmatrix} v_1 cos(\mu t) - w_1 sin(\mu t) \\ v_2 cos(\mu t) - w_2 sin(\mu t) \end{pmatrix}e^{\lambda t} + c_2\begin{pmatrix} v_1 sin(\mu t) + w_1 cos(\mu t) \\ v_2 sin(\mu t) + w_2 cos(\mu t) \end{pmatrix}e^{\lambda t}$$ [redacted]

Case 3a: $\lambda > 0$

Case 3a: $\lambda < 0$

Case 3a: $\lambda = 0$

(a) Node if $q > 0$ and $\Delta \geq 0$; (b) Saddle point if $q < 0$;

(c) Spiral point if $p \neq 0$ and $\Delta < 0$; (d) Center if $p = 0$ and $q > 0$.

Hint: These conclusions can be reached by studying the eigenvalues r_1 and r_2. It may also be helpful to establish, and then to use, the relations $r_1 r_2 = q$ and $r_1 + r_2 = p$.

21. Continuing Problem 20, show that the critical point $(0, 0)$ is

(a) Asymptotically stable if $q > 0$ and $p < 0$;

(b) Stable if $q > 0$ and $p = 0$;

(c) Unstable if $q < 0$ or $p > 0$.

The results of Problems 20 and 21 are summarized visually in Figure 9.1.9.

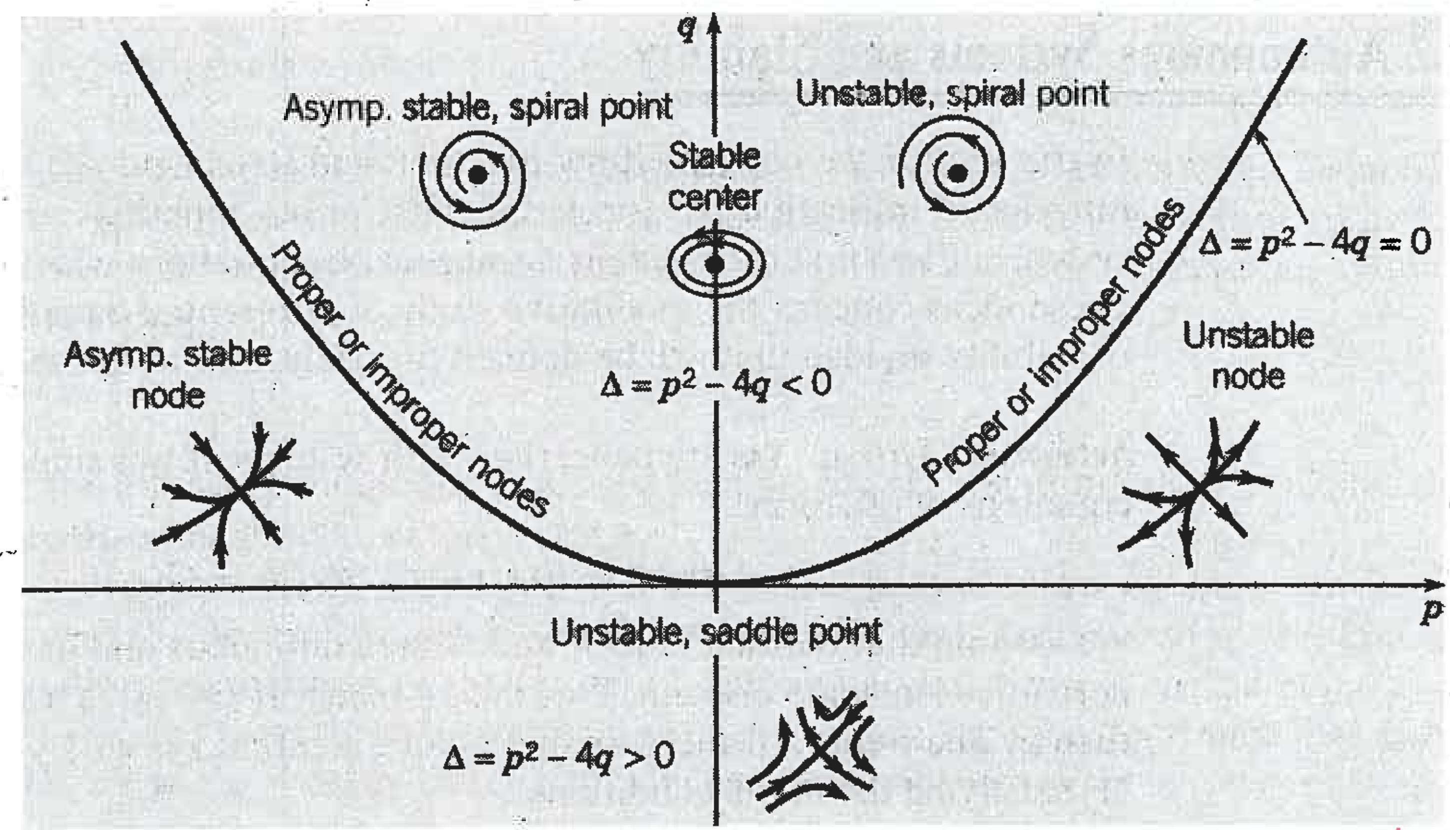


FIGURE 9.1.9 Stability diagram.

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\boxed{\Delta = (a+d)^2 - 4(ad-bc)}$

$p = a + d$

$q = ad - bc$