3.7: Mechanical and Electrical Vibrations

Trig background:

$cos(y \mp x) = cos(x \mp y) = cos(x)cos(y) \pm sin(x)sin(y)$

Let $A = Rcos(\delta)$, $B = Rsin(\delta)$ in

$Acos(\omega_0 t) + Bsin(\omega_0 t)$

$= Rcos(\delta)cos(\omega_0 t) + Rsin(\delta)sin(\omega_0 t)$

$= Rcos(\omega_0 t - \delta)$.

Amplitude $= R$

frequency $= \omega_0$ (measured in radians per unit time).

period $= \frac{2\pi}{\omega_0}$

phase (displacement) $= \delta$

Mechanical Vibrations: RHS

$mu''(t) + \gamma u'(t) + ku(t) = F_{external}, \quad m, \gamma, k \geq 0$

$mg - kL = 0, \quad \boxed{F_{damping}(t) = -\gamma u'(t)}$

m= mass,

k = spring force proportionality constant,

γ = damping force proportionality constant

g = 9.8 m/sec

Electrical Vibrations:

$L\frac{dI(t)}{dt} + RI(t) + \frac{1}{C}Q(t) = E(t), \quad L, R, C \geq 0$ and $I = \frac{dQ}{dt}$

$lQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = E(t)$

L = inductance (henrys),

R = resistance (ohms)

C = capacitance (farads)

$Q(t)$ = charge at time t (coulombs)

$I(t)$ = current at time t (amperes)

$E(t)$ = impressed voltage (volts).

1 volt = 1 ohm · 1 ampere = 1 coulomb / 1 farad = 1 henry · 1 amperes/ 1 second

$$mu''(t) + \gamma u'(t) + ku(t) = 0, \quad m, \gamma, k \geq 0$$

$$r_1, r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}$$

$\gamma^2 - 4km > 0$: $u(t) = Ae^{r_1 t} + Be^{r_2 t}$ ← over damped

$r_1, r_2 < 0$

$\gamma^2 - 4km = 0$: $u(t) = (A + Bt)e^{r_1 t}$ ← critical damping

$-\gamma/2m = r_1 < 0$

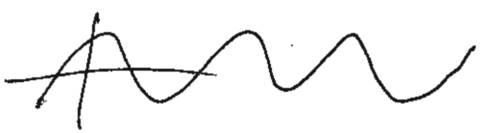
$\gamma^2 - 4km < 0$: $u(t) = e^{-\frac{\gamma t}{2m}}(A\cos\mu t + B\sin\mu t)$ ← underdamped

$= e^{-\frac{\gamma t}{2m}} R\cos(\mu t - \delta)$

where $A = R\cos(\delta)$, $B = R\sin(\delta)$

μ = quasi frequency, $\frac{2\pi}{\mu}$ = quasi period

Note if $\gamma = 0$, then



Critical damping: $\gamma = 2\sqrt{km}$

Overdamped: $\gamma > 2\sqrt{km}$

Amplitude $= \sqrt{2} = R$
frequency $= \sqrt{8}$
period $= 2\pi/\sqrt{8}$
displacement $= 7\pi/4$

Suppose a mass weighs 64 lbs stretches a spring 4 ft. If there is no damping and the spring is stretched an additional foot and set in motion with an upward velocity of $\sqrt{8}$ ft/sec, find the equation of motion of the mass.

$\gamma = 0$

$Weight = mg$: $m = \frac{weight}{g} = \frac{64}{32} = 2$

$mg - kL = 0$ implies $k = \frac{mg}{L} = \frac{64}{4} = 16$

$mu''(t) + \gamma u'(t) + ku(t) = F_{external}$

$[\gamma^2 - 4km < 0$: $u(t) = e^{-\frac{\gamma t}{2m}}(A\cos\mu t + B\sin\mu t)$
Hence $u(t) = A\cos\mu t + B\sin\mu t$ since $\gamma = 0]$.

IVP

$\boxed{2u''(t) + 16u(t) = 0}$

$u''(t) + 8u(t) = 0$, $\quad \boxed{u(0) = 1,\ u'(0) = -\sqrt{8}}$

$r^2 + 8 = 0 \rightarrow r = \pm\sqrt{-8} = \pm i\sqrt{8} = 0 \pm i\sqrt{8}$

$u(t) = c_1 e^{it\sqrt{8}} + c_2 e^{-it\sqrt{8}}$

convert ugly to nice

$u(t) = A\cos\sqrt{8}t + B\sin\sqrt{8}t$

$u(0) = 1$: $1 = A\cos(0) + B\sin(0) = A$

$u'(t) = -\sqrt{8}A\sin\sqrt{8}t + \sqrt{8}B\cos\sqrt{8}t$

$u'(0) = -\sqrt{8}$: $-\sqrt{8} = -\sqrt{8}A\sin(0) + \sqrt{8}B\cos(0)$

$B = -1$ Thus $u(t) = \cos\sqrt{8}t - \sin\sqrt{8}t$

very nice format

$$u(t) = \sqrt{2}\cos\left(\sqrt{8}\,t - \frac{7\pi}{4}\right)$$

Solving polynomial equations:

Ex: $r^3 + r^2 + 3r + 10 = 0$

Plug in $r = \pm 1, \pm 2, \pm 5, \pm 10$ to see if any of these are solutions:

$$(\pm 1)^3 + (\pm 1)^2 + 3(\pm 1) + 10 \neq 0$$

$$(\pm 2)^3 + (\pm 2)^2 + 3(\pm 2) + 10 \ ? =?0$$

$$(-2)^3 + (-2)^2 + 3(-2) + 10 = -8 + 4 - 6 + 10 = 0$$

Thus $(r - (-2))$ is a factor of $r^3 + r^2 + 3r + 10$

Hence $r^3 + r^2 + 3r + 10 = (r+2)(r^2 + \_\_ r + 5)$

$r^3 + r^2 + 3r + 10 = (r+2)(r^2 - r + 5) = 0$

Thus $r = -2, \frac{1 \pm \sqrt{1-20}}{2}$. Thus $r = -2, \frac{1 \pm i\sqrt{19}}{2}$.

In special cases, you can use the unit circle.

Ex: $r^4 + 1 = 0$ implies

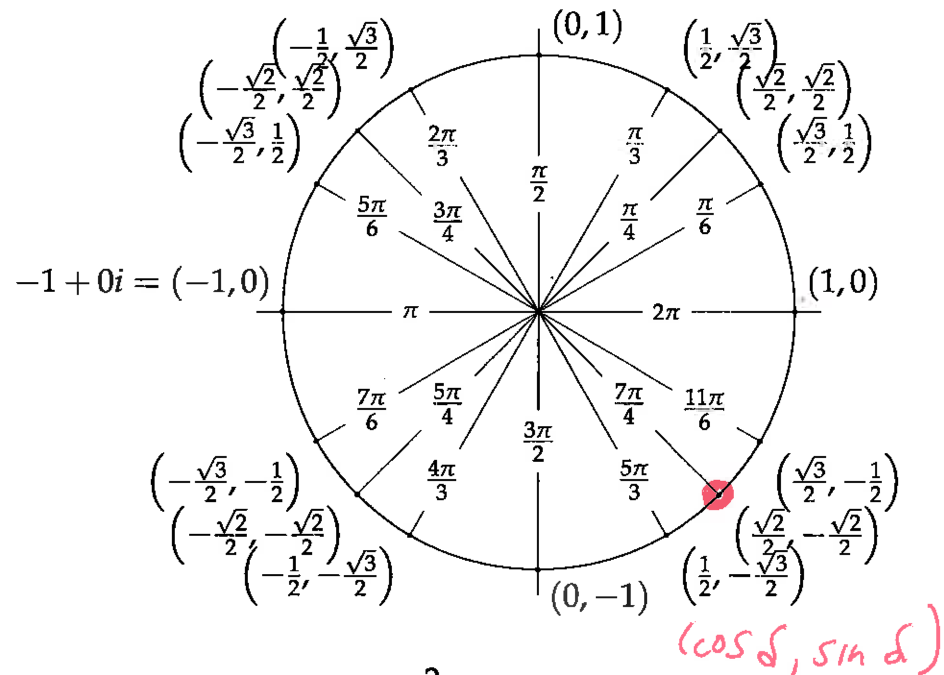
$$r = (-1)^{\frac{1}{4}} = (-1 + 0i)^{\frac{1}{4}} = (e^{i\pi})^{\frac{1}{4}} = (e^{i(\pi + 2\pi k)})^{\frac{1}{4}}$$

$k = 0: \quad e^{\frac{i\pi}{4}} = cos(\frac{i\pi}{4}) + isin(\frac{i\pi}{4}) = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

$k = 1: \quad e^{\frac{3i\pi}{4}} = cos(\frac{3i\pi}{4}) + isin(\frac{3i\pi}{4}) = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$

$k = 2: \quad e^{\frac{5i\pi}{4}} = cos(\frac{5i\pi}{4}) + isin(\frac{5i\pi}{4}) = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

$k = 3: \quad e^{\frac{7i\pi}{4}} = cos(\frac{7i\pi}{4}) + isin(\frac{7i\pi}{4}) = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$



Since $\gamma - \psi$ is a solution to $ay'' + by' + cy = 0$ and

$c_1\phi_1(t) + c_2\phi_2(t)$ is a general solution to
$$ay'' + by' + cy = 0,$$

there exist constants c_1, c_2 such that

$\gamma - \psi =$ ______________________

Thus $\gamma = \psi + c_1\phi_1(t) + c_2\phi_2(t)$.

Thm:
Suppose f_1 is a a solution to $ay'' + by' + cy = g_1(t)$ and f_2 is a a solution to $ay'' + by' + cy = g_2(t)$, then $f_1 + f_2$ is a solution to $ay'' + by' + cy = g_1(t) + g_2(t)$

Proof:

Since f_1 is a solution to $ay'' + by' + cy = g_1(t)$,

Since f_2 is a solution to $ay'' + by' + cy = g_2(t)$,

We will now show that $f_1 + f_2$ is a solution to $ay'' + by' + cy = g_1(t) + g_2(t)$.

Sidenote: The proofs above work even if a, b, c are functions of t instead of constants.

Examples:

Find a suitable form for ψ for the following differential equations:

1.) $y'' - 4y' - 5y = \underline{4e^{2t}}$

2.) $y'' - 4y' - 5y = \underline{4sin(3t)}$

3.) $y'' - 4y' - 5y = \underline{t^2 - 2t + 1}$

4.) $y'' - 5y = 4sin(3t)$

Method of Undetermined ~~a~~ ~~ff~~ Guess

5.) $y'' - 4y' = t^2 - 2t + 1$

6.) $y'' - 4y' - 5y = 4(t^2 - 2t - 1)e^{2t}$

7.) $y'' - 4y' - 5y = 4sin(3t)e^{2t}$

8.) $y'' - 4y' - 5y = 4(t^2 - 2t - 1)sin(3t)e^{2t}$

9.) $y'' - 4y' - 5y = 4sin(3t) + 4sin(3t)e^{2t}$

$r^2 - 4r - 5 = 0 \quad (r-5)(r+1) = 0$

10.) $y'' - \cancel{4y' - 5y}$
$= 4sin(3t)e^{2t} + 4(t^2 - 2t - 1)e^{2t} + t^2 - 2t - 1 + e^{5t}$

$\left[(A\sin 3t + B\cos 3t)e^{2t}\right] + \left[(Ct^2 + Dt + E)e^{2t}\right] + \left[Ft^2 + Gt + H\right] + Kte^{5t}$

11.) $y'' - 4y' - 5y = 4sin(3t) + 5cos(3t)$

12.) $y'' - 4y' - 5y = 4e^{-t}$

$L(f) = ay'' + by' + cy$ is a linear fn

To solve $ay'' + by' + cy = g_1(t) + g_2(t) + ...g_n(t)$ [**]

1.) Find the general solution to $ay'' + by' + cy = 0$:

$$c_1\phi_1 + c_2\phi_2$$

2.) For each g_i, find a solution to $ay'' + by' + cy = g_i$:

$$\psi_i$$

This includes plugging guessed solution into $ay'' + by' + cy = g_i$ to find constant(s).

The general solution to [**] is

$$L(c_1\phi_1 + c_2\phi_2 + \psi_1 + \psi_2 + ...\psi_n) = \underbrace{L(c_1\phi_1 + c_2\phi_2)}_{0} + \overbrace{L(\psi_1)}^{g_1} + \dots + \underbrace{L(\psi_n)}_{g_n}$$

3.) If initial value problem:

Once general solution is known, can solve initial value problem (i.e., use initial conditions to find c_1, c_2).

3.6 Variation of Parameters   Solve $y'' - 2y' + y = e^t ln(t)$

1) Find homogeneous solutions: Solve $y'' - 2y' + y = 0$

Guess: $y = e^{rt}$, then $y' = re^{rt}$, $y'' = r^2e^{rt}$, and

$$r^2e^{rt} - 2re^{rt} + e^{rt} = 0 \text{ implies } r^2 - 2r + 1 = 0$$

$(r-1)^2 = 0$, and hence $r = 1$

General homogeneous solution: $y = c_1e^t + c_2te^t$

since have two linearly independent solutions: $\{e^t, te^t\}$

2.) Find a non-homogeneous solution:

Sect. 3.5 method: Educated guess

Sect. 3.6: **Guess** $y = u_1(t)e^t + u_2(t)te^t$ **and solve for** u_1 **and** u_2

$$u_1(t) = \int \frac{\begin{vmatrix} 0 & \phi_2 \\ 1 & \phi_2' \end{vmatrix}}{\begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}} g(t)dt = -\int \frac{\phi_2(t)g(t)}{W(\phi_1,\phi_2)}dt = -\int \frac{(te^t)(e^tln(t))}{e^{2t}}dt$$

$$= -\int tln(t) = -[\frac{t^2ln(t)}{2} - \int \frac{t}{2}] = -\frac{t^2ln(t)}{2} + \frac{t^2}{4}$$

$$u_2(t) = \int \frac{\begin{vmatrix} \phi_1 & 0 \\ \phi_1' & 1 \end{vmatrix}}{\begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}} g(t)dt = \int \frac{\phi_1(t)g(t)}{W(\phi_1,\phi_2)}dt = \int \frac{(e^t)(e^tln(t))}{e^{2t}}dt$$

$$= \int ln(t) = tln(t) - t$$

$$W(\phi_1, \phi_2) = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix}$$

$u = ln(t)$ $dv = tdt$ $du = \frac{dt}{t}$ $v = \frac{t^2}{2}$

$u = ln(t)$ $dv = dt$ $du = \frac{dt}{t}$ $v = t$

General solution : $y = c_1e^t + c_2te^t + [(-\frac{t^2ln(t)}{2} + \frac{t^2}{4})]e^t + [(tln(t) - t)](te^t)$

which simplifies to $y = c_1e^t + c_2te^t + (\frac{ln(t)}{2} - \frac{3}{4})t^2e^t$

Solve $y'' + p(t)y' + q(t)y = g(t)$ where $y = c_1\phi_1(t) + c_2\phi_2(t)$ is solution to homogeneous equation $y'' + p(t)y' + q(t)y = 0$

Guess $y = u_1(t)\phi_1(t) + u_2(t)\phi_2(t)$

$y = u_1\phi_1 + u_2\phi_2$ implies $y' = u_1\phi_1' + u_1'\phi_1 + u_2\phi_2' + u_2'\phi_2$

Two unknown functions, u_1 and u_2, but only one equation ($y'' + p(t)y' + q(t)y = g(t)$). Thus might be OK to choose 2nd eq'n.

Avoid 2nd derivative in y'': **Choose** $u_1'\phi_1 + u_2'\phi_2 = 0$

$y' = u_1\phi_1' + u_2\phi_2'$ implies $y'' = u_1\phi_1'' + u_1'\phi_1' + u_2\phi_2'' + u_2'\phi_2'$

Plug into $y'' + p(t)y' + q(t)y = g(t)$:

$u_1\phi_1'' + u_1'\phi_1' + u_2\phi_2'' + u_2'\phi_2' + p(u_1\phi_1' + u_2\phi_2') + q(u_1\phi_1 + u_2\phi_2) = g$

$u_1\phi_1'' + u_1'\phi_1' + u_2\phi_2'' + u_2'\phi_2' + pu_1\phi_1' + pu_2\phi_2') + qu_1\phi_1 + qu_2\phi_2 = g$

$u_1\phi_1'' + pu_1\phi_1' + qu_1\phi_1 + u_1'\phi_1' + u_2\phi_2'' + pu_2\phi_2' + qu_2\phi_2 + u_2'\phi_2' = g$

$u_1(\phi_1'' + p\phi_1' + q\phi_1) + u_1'\phi_1' + u_2(\phi_2'' + p\phi_2' + q\phi_2) + u_2'\phi_2' = g$

ϕ_1, ϕ_2 are homogeneous solutions. Thus $\phi_i'' + p\phi_i' + q\phi_i = 0$.

Hence $u_1(0) + u_1'\phi_1' + u_2(0) + u_2'\phi_2' = g$

Thus we have 2 eqns to find 2 unknowns, the functions u_1 and u_2:

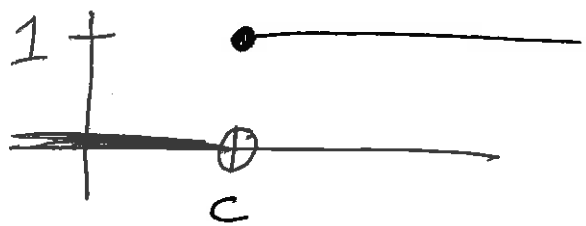
$$\begin{aligned} u_1'\phi_1 + u_2'\phi_2 &= 0 \\ u_1'\phi_1' + u_2'\phi_2' &= g \end{aligned} \quad \text{implies} \quad \begin{bmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} \begin{bmatrix} 0 \\ g \end{bmatrix}$$

Cramer's rule: $u_1'(t) = \frac{\begin{vmatrix} 0 & \phi_2 \\ g & \phi_2' \end{vmatrix}}{\begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}}$ and $u_2'(t) = \frac{\begin{vmatrix} \phi_1 & 0 \\ \phi_1' & g \end{vmatrix}}{\begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}}$

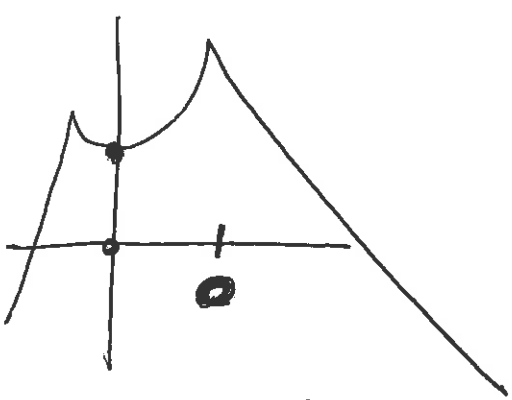
6.3: Step functions.

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$

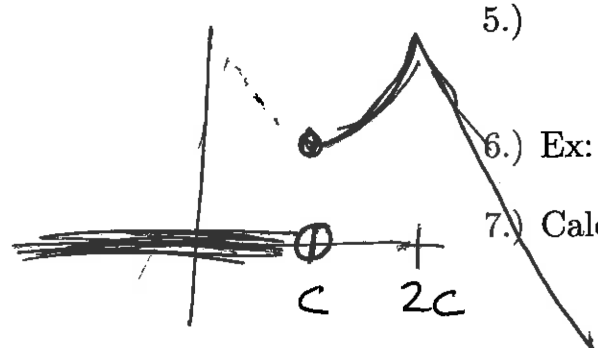
1.) Graph $u_c(t)$:



2.) Given f, graph $u_c(t)f(t-c)$:



$y = f(t)$



$y = u_c(t) f(t-c)$

chop   shift right

3.) Calculate $\mathcal{L}(u_c(t)f(t-c))$ in terms of $\mathcal{L}(f(t))$:

$$\mathcal{L}(u_c f(t-c)) = \int_0^\infty e^{-st} u_c(t) f(t-c)\, dt$$

$$= \int_c^\infty e^{-st} f(t-c)\, dt$$

Example: Find the LaPlace transform of

4.) $$g(t) = \begin{cases} 0 & t < 3 \\ e^{t-3} & t \geq 3 \end{cases}$$

5.) $$f(t) = \begin{cases} 0 & t < 3 \\ 5 & 3 \leq t < 4 \\ t-5 & t \geq 4 \end{cases}$$

6.) Ex: Find the inverse Laplace transform of $\frac{e^{-8s}}{s^3}$

7.) Calculate $\mathcal{L}(e^{ct}f(t))$ in terms of $F(s) = \mathcal{L}(f(t))$

8.) Example: Use formula 6 (p. 317) to find the inverse LaPlace transform of $\frac{s-c}{(s-c)^2+a^2}$.