Solving for y

The LaPlace Transform is a method to change a differential equation to a linear equation.

Example: Solve $y'' + 3y' + 4y = 0$, $y(0) = 5$, $y'(0) = 6$

1.) Take the LaPlace Transform of both sides of the equation:

$\mathcal{L}(y'' + 3y' + 4y) = \mathcal{L}(0)$

2.) Use the fact that the LaPlace Transform is linear:

$\mathcal{L}(y'') + 3\mathcal{L}(y') + 4\mathcal{L}(y) = 0$

$\mathcal{L}$ is linea
$\mathcal{L}(0) = 0$

3.) Use thm to change this equation into an algebraic equation:

$[s^2\mathcal{L}(y) - sy(0) - y'(0)] + 3[s\mathcal{L}(y) - y(0)] + 4\mathcal{L}(y) = 0$

3.5) Substitute in the initial values:

$s^2\mathcal{L}(y) - 5s - 6 + 3[s\mathcal{L}(y) - 5] + 4\mathcal{L}(y) = 0$

4.) Solve the algebraic equation for $\mathcal{L}(y)$

$s^2\mathcal{L}(y) - 5s - 6 + 3s\mathcal{L}(y) - 15 + 4\mathcal{L}(y) = 0$

$[s^2 + 3s + 4]\mathcal{L}(y) = 5s + 21$

$\mathcal{L}(y) = \frac{5s+21}{s^2+3s+4}$

Some algebra implies $\mathcal{L}(y) = \frac{5s+21}{s^2+3s+4}$

5.) Solve for y by taking the inverse LaPlace transform of both sides (use a table):

$\mathcal{L}^{-1}(\mathcal{L}(y)) = \mathcal{L}^{-1}(\frac{5s+21}{s^2+3s+4})$

$y = \mathcal{L}^{-1}(\frac{5s+21}{s^2+3s+4})$

Find the inverse LaPlace transform of $\frac{5s+21}{s^2+3s+4}$

Look at the denominator first to determine if it is of the form $s^2 \pm a^2$ or $(s-a)^{n+1}$ or $(s-a)^2 + b^2$ OR if you should factor and use partial fractions

$s^2 + 3s + 4$: $b^2 - 4ac = 3^2 - 4(1)(4) = 9 - 16 < 0$

Hence s^2+3s+4 does not factor over the reals. Hence to avoid complex numbers, we won't factor it.

s^2+3s+4 is not an s^2-a^2 or an s^2+a^2 or an $(s-a)^2$, so it must be an $(s-a)^2+b^2$.

Hence we will complete the square:

$s^2 + 3s + \_\_\_ - \_\_\_ + 4 = (s + \_\_\_)^2 - \_\_\_ + 4$

Hence $\frac{5s+21}{s^2+3s+4} = \frac{5s+21}{(s+\frac{3}{2})^2+\frac{7}{4}}$

Must now consider the numerator. We need it to look like $s - a = s + \frac{3}{2}$ or $b = \sqrt{\frac{7}{4}}$ in order to use $\mathcal{L}^{-1}(\frac{s-a}{(s-a)^2+b^2}) = e^{at}cosbt$ and/or $\mathcal{L}^{-1}(\frac{b}{(s-a)^2+b^2}) = e^{at}sinbt$

$5s + 21 = 5(s + \frac{3}{2}) - \frac{15}{2} + 21 = 5(s + \frac{3}{2}) - \frac{27}{2}$

$= 5(s + \frac{3}{2}) - [\frac{27}{2}\sqrt{\frac{4}{7}}]\sqrt{\frac{7}{4}} = 5(s + \frac{3}{2}) - [\frac{27}{\sqrt{7}}]\sqrt{\frac{7}{4}}$

Hence $\frac{5s+21}{s^2+3s+4} = \frac{5(s+\frac{3}{2}) - [\frac{27}{\sqrt{7}}]\sqrt{\frac{7}{4}}}{(s+\frac{3}{2})^2+\frac{7}{4}}$

$= 5[\frac{s+\frac{3}{2}}{(s+\frac{3}{2})^2+\frac{7}{4}}] - \frac{27}{\sqrt{7}}[\frac{\sqrt{\frac{7}{4}}}{(s+\frac{3}{2})^2+\frac{7}{4}}]$

Thus $\mathcal{L}^{-1}(\frac{5s+21}{s^2+3s+4}) = \mathcal{L}^{-1}(5[\frac{s+\frac{3}{2}}{(s+\frac{3}{2})^2+\frac{7}{4}}] - \frac{27}{\sqrt{7}}[\frac{\sqrt{\frac{7}{4}}}{(s+\frac{3}{2})^2+\frac{7}{4}}])$

$= 5\mathcal{L}^{-1}(\frac{s+\frac{3}{2}}{(s+\frac{3}{2})^2+\frac{7}{4}}) - \frac{27}{\sqrt{7}}\mathcal{L}^{-1}(\frac{\sqrt{\frac{7}{4}}}{(s+\frac{3}{2})^2+\frac{7}{4}})$

$= 5e^{-\frac{3}{2}t}cos\sqrt{\frac{7}{4}}t - \frac{27}{\sqrt{7}}e^{-\frac{3}{2}t}sin\sqrt{\frac{7}{4}}t$

Hence $y(t) = 5e^{-\frac{3}{2}t}cos\sqrt{\frac{7}{4}}t - \frac{27}{\sqrt{7}}e^{-\frac{3}{2}t}sin\sqrt{\frac{7}{4}}t.$

$0 \cdot \mathcal{L}(0) = 0$

$\mathcal{L}$ is a linear fn $\Rightarrow$ $\mathcal{L}(0) = 0$

TABLE 6.2.1 Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	Notes
1. 1	$\frac{1}{s}, \quad s > 0$	Sec. 6.1; Ex. 4
2. e^{at}	$\frac{1}{s-a}, \quad s > a$	Sec. 6.1; Ex. 5
3. t^n, $n =$ positive integer	$\frac{n!}{s^{n+1}}, \quad s > 0$	Sec. 6.1; Prob. 27
4. t^p, $p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$	Sec. 6.1; Prob. 27
5. $\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Ex. 6
6. $\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Prob. 6
7. $\sinh at$	$\frac{a}{s^2-a^2}, \quad s > \lvert a \rvert$	Sec. 6.1; Prob. 8
8. $\cosh at$	$\frac{s}{s^2-a^2}, \quad s > \lvert a \rvert$	Sec. 6.1; Prob. 7
9. $e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 13
10. $e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 14
11. $t^n e^{at}$, $n =$ positive integer	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$	Sec. 6.1; Prob. 18
12. $u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$	Sec. 6.3
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$	Sec. 6.3
14. $e^{ct}f(t)$	$F(s-c)$	Sec. 6.3
15. $f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right), \quad c > 0$	Sec. 6.3; Prob. 19
16. $\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	Sec. 6.6
17. $\delta(t-c)$	e^{-cs}	Sec. 6.5
18. $f^{(n)}(t)$	$s^n F(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	Sec. 6.2
19. $(-t)^n f(t)$	$F^{(n)}(s)$	Sec. 6.2; Prob. 28