5.3: Series solutions near an ordinary point, part II

A power series solution exists in a neighborhood of x_0 when the solution is analytic at x_0. I.e, the solution is of the form $y = \Sigma_{n=0}^{\infty} a_n (x - x_0)^n$ where this series has a nonzero radius of convergence about x_0.

That is $f(x) = \Sigma_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$ for x near x_0.

Thus there are constants $a_n = \frac{f^{(n)}(x_0)}{n!}$ such that,

$$f(x) = \Sigma_{n=0}^{\infty} a_n (x - x_0)^n.$$

When do we know an analytic solution exists? I.e, when is this method guaranteed to work?

Special case: $P(x)y'' + Q(x)y' + R(x)y = 0$

Then $y''(x) = -[\frac{Q}{P}y' + \frac{R}{P}y]$

$y'''(x) = -[(\frac{Q}{P})'y' + \frac{Q}{P}y'' + \frac{R}{P}'y + \frac{R}{P}y']$

If $f(x) = \Sigma_{n=0}^{\infty} a_n (x - x_0)^n$ is a solution where $a_n = \frac{f^{(n)}(x_0)}{n!}$, then $a_0 = f(x_0), a_1 = f'(x_0)$

$2!a_2 = f''(x_0) = -[\frac{Q}{P}f'(x_0) + \frac{R}{P}f(x_0)] = -[\frac{Q}{P}a_1 + \frac{R}{P}a_0]$

$3!a_3 = f'''(x_0) = -[(\frac{Q}{P})'f'(x_0) + \frac{Q}{P}f''(x_0) + \frac{R}{P}'f(x_0) + \frac{R}{P}f'(x_0)]$

To find a_n we could continue taking derivative including derivatives of $\frac{Q}{P}$ and $\frac{R}{P}$ (but much easier to plug series into equation – ie 5.2 method).

Definition: The point x_0 is an *ordinary point* of the ODE,

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

if $\frac{Q}{P}$ and $\frac{R}{P}$ are analytic at x_0. If x_0 is not an ordinary point, then it is a *singular point.*

Theorem 5.3.1: If x_0 is an ordinary point of the ODE $P(x)y'' + Q(x)y' + R(x)y = 0$, then the general solution to this ODE is

$$y = \Sigma_{n=0}^{\infty} a_n (x - x_0)^n = a_0\phi_0(x) + a_1\phi_1(x)$$

where ϕ_i are power series solutions that are analytic at x_0. The solutions ϕ_0, ϕ_1 form a fundamental set of solutions. The radius of convergence for each of these series solutions is at least as large as the minimum radii of convergence of the series for $\frac{Q}{P}$ and $\frac{R}{P}$.

Theorem: If P and Q are polynomial functions with no common factors, then $y = Q(x)/P(x)$ is analytic at x_0 if and only if $P(x_0) \neq 0$. Moreover the radius of convergence of $Q(x)/P(x)$ is

$$min\{||x_0 - x|| \mid x \in \mathbf{C}, P(x) = 0\}$$

where $||x_0 - x||$ = distance from x_0 to x in the complex plane.

Ex: $x(x+1)y'' + \frac{x^2}{x^2+1}y' + \frac{x}{x-2}y = 0$

$y'' + \frac{x}{(x^2+1)(x+1)}y' + \frac{1}{(x-2)(x+1)}y = 0$

Then $x_0 = -1, 2$ are singular points. All other points are ordinary points.

The zeros of the denominators are $x = \pm i, -1, 2$

Radius of convergence for the series solution to this ODE about the point x_0 if $x_0 \neq -1, 2$ is at least as large as minimum$\{\sqrt{x_0^2 + (\pm 1)^2}, |x_0 - (-1)|, |x_0 - 2|\}$

If $x_0 = 0$, radius of convergence ≥ 1

If $x_0 = -3$, radius of convergence ≥ 2

If $x_0 = 3$, radius of convergence ≥ 1

If $x_0 = \frac{1}{3}$, radius of convergence $\geq \sqrt{(\frac{1}{3})^2 + (\pm 1)^2} = \frac{\sqrt{10}}{3}$

$$\textcircled{1}\ y'' + \frac{\alpha y'}{x} + \frac{\beta y}{x^2} = 0$$

To determine ordinary vs singular pts

5.4: Euler equation: $\boxed{x^2y'' + \alpha xy' + \beta y = 0}$

$\quad x^2 \quad x^2 \quad x^2 \quad x^2$

Let $L(y) = x^2y'' + \alpha xy' + \beta y$

Recall that L is a linear function and if f is a solution to the euler equation, then $L(f) = 0$.

Note that if $x \neq 0$, then x is an ordinary point and if $x = 0$, then x is a singular point.

$\Rightarrow$ singular pt

Suppose $x > 0$. Claim $L(x^r) = 0$ for some value of r

$y = x^r$, $y' = rx^{r-1}$, $y'' = r(r-1)x^{r-2}$

$x^2y'' + \alpha xy' + \beta y = 0$

$x^2r(r-1)x^{r-2} + \alpha xrx^{r-1} + \beta x^r = 0$

$(r^2 - r)x^r + \alpha rx^r + \beta x^r = 0$

$x^r[r^2 - r + \alpha r + \beta] = 0$

$x^r[r^2 + (\alpha - 1)r + \beta] = 0$

Thus x^r is a solution iff $r^2 + (\alpha - 1)r + \beta = 0$

Thus $r = \frac{-(\alpha-1)\pm\sqrt{(\alpha-1)^2-4\beta}}{2}$

Suppose $x < 0$. Claim $L((-x)^r) = 0$ for some value of r

$y = (-x)^r$, $y' = -r(-x)^{r-1}$, $y'' = r(r-1)(-x)^{r-2}$

$x^2y'' + \alpha xy' + \beta y = 0$

$x^2r(r-1)(-x)^{r-2} - \alpha xr(-x)^{r-1} + \beta(-x)^r = 0$

$(r^2 - r)(-x)^r + \alpha r(-x)^r + \beta(-x)^r = 0$

$(-x)^r[r^2 - r + \alpha r + \beta] = 0$

$(-x)^r[r^2 + (\alpha - 1)r + \beta] = 0$

Thus $(-x)^r$ is a solution iff $r^2 + (\alpha - 1)r + \beta = 0$

Thus $r = \frac{-(\alpha-1)\pm\sqrt{(\alpha-1)^2-4\beta}}{2}$

Recall $|x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases}$

Thus $|x|^r = \begin{cases} x^r & \text{if } x > 0 \\ (-x)^r & \text{if } x < 0 \end{cases}$

Thus if $r = \frac{-(\alpha-1)\pm\sqrt{(\alpha-1)^2-4\beta}}{2}$, then $y = |x|^r$ is a solution to Euler's equation for $x \neq 0$.

Case 1. 2 real distinct roots, r_1, r_2:
General solution is $y = c_1|x|^{r_1} + c_2|x|^{r_2}$.

Case 2: 2 complex solutions $r_i = \lambda \pm i\mu$:

Convert solution to form without complex numbers.

Note $|x|^{\lambda\pm i\mu} = e^{ln(|x|^{\lambda\pm i\mu})} = e^{(\lambda\pm i\mu)ln|x|} = e^{\lambda ln|x|}e^{i(\pm\mu ln|x|)}$

$= |x|^\lambda[cos(\pm\mu ln|x|) + isin(\pm\mu ln|x|)]$

$= |x|^\lambda[cos(\mu ln|x|) \pm isin(\mu ln|x|)]$

Case 3: 1 repeated root: Find 2nd solution.

Section 5.4 continued

Solve $x^2y'' - 2xy' = 0$ (*).

We could solve by letting $v = y'$, but we will instead use 5.4 methods

Note x is an ordinary point iff $x \neq 0$ $\quad (y'' - \frac{2}{x}y' = 0.)$
$x = 0$ is a singular point.

Note $x^2x^{r-2}r(r-1) - 2xx^{r-1}r = 0$ implies $r^2 - r - 2r = 0$ and recall $y = (-x)^r$ gives same equation for r as $y = x^r$.

Thus $y = |x|^r$ implies $r^2 + (\alpha - 1)r + \beta = r^2 - 3r + 0 = r(r-3) = 0$

Thus $r = 0, 3$. Thus $y = |x|^0 = 1$ and $y = |x|^3$ are solutions to (*)

Since (*) is a linear equation, the general solution is $y = c_1 + c_2|x|^3$.

Note an equivalent general solution is $y = k_1 + k_2x^3$.

Both forms are valid for all x.

When is a unique solution to the following initial value problem guaranteed?

$$x^2y'' - 2xy' = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_1 \ (^{**})$$

$$y'' - \tfrac{2}{x}y' = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y_1$$

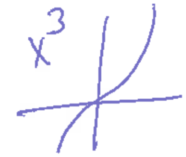
Since $\frac{2}{x}$ and the zero constant function are continuous on $(-\infty, 0) \cup (0, \infty)$,

(**) has a unique solution for $t_0 < 0$ and this solution exists on $(-\infty, 0)$.

(**) has a unique solution for $t_0 > 0$ and this solution exists on $(0, \infty)$.

There are an infinite number of solutions for $y(0) = a$, $y'(0) = 0$.

Side note:



How is x^r defined:

If n is a positive integer: $x^n = \overbrace{x \cdot x \cdot \ldots \cdot x}^{n \text{ times}}$

If m is a positive integer: If $f(x) = x^m$, then $f^{-1}(x) = x^{\frac{1}{m}}$ and $x^{\frac{n}{m}} = (x^n)^{\frac{1}{m}}$

Let $r \geq 0$. Let r_n be any sequence consisting of positive rational numbers such that $lim_{n\to\infty} r_n = r$. Then

$$x^r = lim_{n\to\infty} x^{r_n}.$$

See more advanced class for why the above is well-defined.

If $r < 0$, then $x^r = x^{-r}$.

If x is a real number, when is x^r a real number?

$x^n = x \cdot x \cdot \ldots \cdot x$ is a real number when n is a positive integer.

If $f(x) = x^n$, then the image of $f = \begin{cases} \text{real numbers} & n \text{ odd} \\ [0, \infty) & n \text{ even} \end{cases}$

(domain of f is $\mathbb{R}$

Thus if $f^{-1}(x) = x^{\frac{1}{n}}$ is real-valued, then

the domain of f^{-1} is $\begin{cases} \text{real numbers} & n \text{ odd} \\ [0, \infty) & n \text{ even} \end{cases}$

This is why break into case $x > 0$ & $x < 0$ for Euler's eqns

In complex analysis, $\left(\frac{1+i\sqrt{3}}{2}\right)^3 = -1$, $(-1)^3 = -1$, $\left(\frac{1-i\sqrt{3}}{2}\right)^3 = -1$

Recall $\left(e^{\frac{i\pi}{3}}\right)^3 = (cos\frac{\pi}{3} + isin\frac{\pi}{3})^3 = -1$

Complex numbers are also roots of unity:

$$\left(e^{\frac{2i\pi}{3}}\right)^3 = 1 \quad \left(e^{\frac{-2i\pi}{3}}\right)^3 = 1, \quad (1)^3 = 1$$

a different side note for later

In ch 5, want $x \in \mathbb{R}$

but r can be complex

$x^{a+bi} = e^{\ln x^{a+bi}} = e^{(a+bi)\ln x}$