normal solutions and two will be the normal solution each multiplied by t.

From: Paul's online notes

The general solution is,

$$y(t) = c_1 + c_2 \mathbf{e}^{-3t}\cos(5t) + c_3\mathbf{e}^{-3t}\sin(5t) + c_4\, t\,\mathbf{e}^{-3t}\cos(5t) + c_5\, t\,\mathbf{e}^{-3t}\sin(5t)$$

Let's now work an example that contains all three of the basic cases just to say that we that we've got one work here.

Example 4 Solve the following differential equation. Guess $y = e^{rt}$

$$y^{(5)} - 15y^{(4)} + 84y^{(3)} - 220y'' + 275y' - 125y = 0$$

$r^5 e^{rt}$ $r^4 e^{rt}$ $r^3 e^{rt}$ $r^2 e^{rt}$ $r e^{rt}$ e^{rt}

Solution

The characteristic equation is

$$r^5 - 15r^4 + 84r^3 - 220r^2 + 275r - 125 = (r-1)(r-5)^2\underbrace{(r^2 - 4r + 5)}_{=0} = 0$$

$r = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$

In this case we've got one real distinct root, $r = 1$, and $\boxed{\text{double root, } r = 5}$, and a pair of complex roots, $r = 2 \pm i$ that only occur once.

The general solution is then,

$$y(t) = c_1\mathbf{e}^{t} + c_2\mathbf{e}^{5t} + c_3\, t\,\mathbf{e}^{5t} + c_4\mathbf{e}^{2t}\cos(t) + c_5\mathbf{e}^{2t}\sin(t)$$

$y(t_0) = y_0$
$y'(t_0) = y_1$
$y''(t_0) = y_2$
$y'''(t_0) = y_3$
$y^{iv}(t_0) = y_4$

We've got one final example to work here that on the surface at least seems almost too easy. The problem here will be finding the roots as well see.

Example 5 Solve the following differential equation.

$$y^{(4)} + 16y = 0$$

$r^4 + 16 = 0$

Solution

The characteristic equation is

Derivation of general solutions:

If $b^2 - 4ac > 0$ we guessed e^{rt} is a solution and noted that any linear combination of solutions is a solution to a homogeneous linear differential equation.

Section 3.3: If $b^2 - 4ac < 0$, :

Changed format of $y = c_1e^{r_1t} + c_2e^{r_2t}$ to linear combination of real-valued functions instead of complex valued functions by using Euler's formula:

$$e^{it} = cos(t) + isin(t)$$

Hence $e^{(d+in)t} = e^{dt}e^{int} = e^{dt}[cos(nt) + isin(nt)]$

Let $r_1 = d + in$, $r_2 = d - in$

$$y = c_1e^{r_1t} + c_2e^{r_2t}$$
$$= c_1e^{dt}[cos(nt)+isin(nt)]+c_2e^{dt}[cos(-nt)+isin(-nt)]$$
$$= c_1e^{dt}cos(nt)+ic_1e^{dt}sin(nt)+c_2e^{dt}cos(nt)-ic_2e^{dt}sin(nt)$$
$$=(c_1 + c_2)e^{dt}cos(nt) + i(c_1 - c_2)e^{dt}sin(nt)$$
$$= k_1e^{dt}cos(nt) + k_2e^{dt}sin(nt)$$

Section 3.4: If $b^2 - 4ac = 0$, then $r_1 = r_2$.
Hence one solution is $y = e^{r_1t}$ Need second solution.

If $y = e^{rt}$ is a solution, $y = ce^{rt}$ is a solution.

How about $y = v(t)e^{rt}$?

$$y' = v'(t)e^{rt} + v(t)re^{rt}$$
$$y'' = v''(t)e^{rt} + v'(t)re^{rt} + v'(t)re^{rt} + v(t)r^2e^{rt}$$
$$= v''(t)e^{rt} + 2v'(t)re^{rt} + v(t)r^2e^{rt}$$

$$ay'' + by' + cy = 0$$

$$a(v''e^{rt}+2v're^{rt}+vr^2e^{rt})+b(v'e^{rt}+vre^{rt})+cve^{rt} = 0$$

$$a(v''(t)+2v'(t)r+v(t)r^2)+b(v'(t)+v(t)r)+cv(t) = 0$$

$$av''(t)+2av'(t)r+av(t)r^2+bv'(t)+bv(t)r+cv(t) = 0$$

$$av''(t) + (2ar + b)v'(t) + (ar^2 + br + c)v(t) = 0$$

$$av''(t) + (2a(\tfrac{-b}{2a}) + b)v'(t) + 0 = 0$$

since $ar^2 + br + c = 0$ and $r = \frac{-b}{2a}$

$av''(t) + (-b + b)v'(t) = 0.$ Thus $av''(t) = 0.$

Hence $v''(t) = 0$ and $v'(t) = k_1$ and $v(t) = k_1t + k_2$

Hence $v(t)e^{r_1t} = (k_1t + k_2)e^{r_1t}$ is a soln

Thus te^{r_1t} is a nice second solution.

Hence general solution is $y = c_1e^{r_1t} + c_2te^{r_1t}$

Recall $\phi_1, ..., \phi_n$ are linearly independent iff $c_1 = ... = c_n = 0$ is the only solution to $c_1\phi_1 + ... + c_n\phi_n = 0$.

If ϕ_i are functions of t, then $\mathbf{0}$ is the constant function, $\mathbf{0}(t) = 0$ for all t. Thus $c_1\phi_1(t) + ... + c_n\phi_n(t) = 0$ for all t.

Hence $c_1\phi_1^{(k)}(t) + ... + c_n\phi_n^{(k)}(t) = 0$ for all t, k if derivatives exist.

Thus $\phi_1, ..., \phi_n$ are linearly independent iff for any given f, $c_1\phi_1(t) + ... + c_n\phi_n(t) = 0$ has a unique solution (that works for all t).

iff the following system of equations has a unique solution

$$c_1\phi_1(t) + c_2\phi_2(t) + ... + c_n\phi_n(t) = 0$$
$$c_1\phi_1'(t) + c_2\phi_2'(t) + ... + c_n\phi_n'(t) = 0$$
$$\vdots$$
$$c_1\phi_1^{(n-1)}(t) + c_2\phi_2^{(n-1)}(t) + ... + c_n\phi_n^{(n-1)}(t) = 0$$

iff the following system of equations has a unique solution

$$\begin{bmatrix} \phi_1(t) & \phi_2(t) & \dots & \phi_n(t) \\ \phi_1'(t) & \phi_2'(t) & \dots & \phi_n'(t) \\ & \vdots & & \\ \phi_1^{(n-1)}(t) & \phi_2^{(n-1)}(t) & \dots & \phi_n^{(n-1)}(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Note this equation has a unique solution if and only if for some t_0

$$det \begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) & \dots & \phi_n(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) & \dots & \phi_n'(t_0) \\ & \vdots & & \\ \phi_1^{(n-1)}(t_0) & \phi_2^{(n-1)}(t_0) & \dots & \phi_n^{(n-1)}(t_0) \end{bmatrix} \neq 0$$

iff $\quad W(\phi_1, \phi_2, ..., \phi_n)(t_0) \neq 0$,

Example: Determine if $\{1+2t, 5+4t^2, 6-8t+8t^2\}$ are linearly independent:

Method 1: Solve $c_1(1+2t) + c_2(5+4t^2) + c_3(6-8t+8t^2) = 0$

Or equivalently, solve $c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} 6 \\ -8 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Or equivalently, solve $\begin{bmatrix} 1 & 5 & 6 \\ 2 & 0 & -8 \\ 0 & 4 & 8 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Method 2: Check the Wronskian

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nth order LINEAR differential equation: existence & uniqueness

Theorem 4.1.1: If $p_i : (a,b) \to R$, $i = 1, ..., n$ and $g : (a,b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a,b) \to R$ that satisfies the initial value problem

1 $\quad y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = g(t)$,

$$y(t_0) = y_0, \quad y'(t_0) = y_1, ..., y^{(n-1)}(t_0) = y_{n-1}$$ 2 need n initial values at t_0

Proof: We proved the case $n = 1$ using an integrating factor. When $n > 1$, see more advanced textbook.

Claim: If p_i are continuous on (a,b), if $\phi_1, ..., \phi_n$ are linearly independent solutions to

$$y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = 0,$$

then $\{\phi_1, ..., \phi_n\}$ is a basis for the solution set to this differential equation.

Theorem 4.1.2: If p_i are continuous on (a,b), suppose that ϕ_i, $i = 1, ..., n$ are solutions to $y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = 0$. If $W(\phi_1, \phi_2, ..., \phi_n)(t_0) \neq 0$, for some $t_0 \in (a,b)$, then any solution to this homogeneous linear differential equation can be written as

$$y = c_1\phi_1 + c_2\phi_2 + ... + c_n\phi_n \text{ for some constants } c_i.$$

Defn: The $\phi_1, ..., \phi_n$ are called a fundamental set of solutions to

$$y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = 0.$$

Theorem: Given any nth order homogeneous linear differential equation, there exist a set of n functions which form a fundamental set of solutions.

$$\underbrace{y^{(n)} + p_1(t) y^{(n-1)} + \cdots + p_n(t) y}_{L(y)} = 0$$

4.1: General Theory of nth Order Linear Equations

Thm: $L(y) = y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y$ is a linear function.

Proof: Let a, b be real numbers.

$L(af + bg)$

$= (af+bg)^{(n)} + p_1(t)(af+bg)^{(n-1)} + ... + p_{n-1}(t)(af+bg)' + p_n(t)(af+bg)$

$= af^{(n)} + bg^{(n)} + p_1(af^{(n-1)} + bg^{(n-1)}) + ... + p_{n-1}(af' + bg') + p_n(af+bg)$

$= af^{(n)} + p_1 af^{(n-1)} + ... + p_{n-1}af' + p_n af + bg^{(n)} + p_1 bg^{(n-1)} + ... + p_{n-1}bg' + p_n bg$

$= a[f^{(n)} + p_1 f^{(n-1)} + ... + p_{n-1}f' + p_n f] + b[g^{(n)} + p_1 g^{(n-1)} + ... + p_{n-1}g' + p_n g]$

$= aL(f) + bL(g)$

Theorem: If ϕ_i, $i = 1, ..., n$ are solutions to a homogeneous linear differential equation (i.e., $y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = 0$ (*)), then $c_1\phi_1 + c_2\phi_2 + ... + c_n\phi_n$ is also a solution to this linear differential equation.

Pf: Since ϕ_i, $i = 1, ..., n$ are solutions to (*), $L(\phi_i) = 0$ for $i = 1, ..., n$.
Thus $L(c_1\phi_1 + c_2\phi_2 + ... + c_n\phi_n) = c_1L(\phi_1) + c_2L(\phi_2) + ... + c_nL(\phi_n) = 0$.
Thus $c_1\phi_1 + c_2\phi_2 + ... + c_n\phi_n$ is also a solution to (*).

Solve: $y'' + y = 0$, $y(0) = -1$, $y'(0) = -3$

$r^2 + 1 = 0$ implies $r^2 = -1$. Thus $r = \pm i$.

Since $r = 0 \pm 1i$, $y = k_1 cos(t) + k_2 sin(t)$. Then $y' = -k_1 sin(t) + k_2 cos(t)$

$y(0) = -1$: $\quad -1 = k_1 cos(0) + k_2 sin(0)$ implies $-1 = k_1$

$y'(0) = -3$: $\quad -3 = -k_1 sin(0) + k_2 cos(0)$ implies $-3 = k_2$

Thus IVP solution: $y = -cos(t) - 3sin(t)$

When does the following IVP have a unique solution:

IVP: $y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = 0,$
$y(t_0) = y_0,\ y'(t_0) = y_1, ..., y^{(n-1)}(t_0) = y_{n-1}.$

Suppose $y = c_1\phi_1(t) + c_2\phi_2(t) + ... + c_n\phi_n(t)$ is a solution to this IVT Then

$y(t_0) = y_0$: $\quad y_0 = c_1\phi_1(t_0) + c_2\phi_2(t_0) + ... + c_n\phi_n(t_0)$

$y'(t_0) = y_1$: $\quad y_1 = c_1\phi_1'(t_0) + c_2\phi_2'(t_0) + ... + c_n\phi_n'(t_0)$

$\vdots$

$y^{(n-1)}(t_0) = y_{n-1}$: $\quad y_{n-1} = c_1\phi_1^{(n-1)}(t_0) + c_2\phi_2^{(n-1)}(t_0) + ... + c_n\phi_n^{(n-1)}(t_0)$

Solve for c_i

To find IVP solution, need to solve above system of equations for the unknowns c_i, $i = 1, ..., n$.

Note the IVP has a unique solution if and only if the above system of equations has a unique solution for the c_i.

Note that in these equations the c_i are the unknowns and the $y_i, \phi_i(t_0), ...\phi_i^{(n-1)}(t_0)$, are the constants.
We can translate this linear system of equations into matrix form:

$$\begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) & ... & \phi_n(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) & ... & \phi_n'(t_0) \\ & \vdots & & \\ \phi_1^{(n-1)}(t_0) & \phi_2^{(n-1)}(t_0) & ... & \phi_n^{(n-1)}(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix}$$

Note this equation has a unique solution if and only if

$$det \begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) & ... & \phi_n(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) & ... & \phi_n'(t_0) \\ & \vdots & & \\ \phi_1^{(n-1)}(t_0) & \phi_2^{(n-1)}(t_0) & ... & \phi_n^{(n-1)}(t_0) \end{bmatrix} \neq 0$$

Definition: The Wronskian of the functions, $\phi_1, \phi_2, ..., \phi_n$ is

$$W(\phi_1, \phi_2, ..., \phi_n) = det \begin{bmatrix} \phi_1(t) & \phi_2(t) & ... & \phi_n(t) \\ \phi_1'(t) & \phi_2'(t) & ... & \phi_n'(t) \\ & \vdots & & \\ \phi_1^{(n-1)}(t) & \phi_2^{(n-1)}(t) & ... & \phi_n^{(n-1)}(t) \end{bmatrix}$$

Theorem: Suppose that ϕ_i, $i = 1, ..., n$ are solutions to

$$y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = 0.$$

If $W(\phi_1, \phi_2, ..., \phi_n)(t_0) \neq 0$, then there is a unique choice of constants c_i such that $c_1\phi_1 + c_2\phi_2 + ... + c_n\phi_n$ satisfies this homogeneous linear differential equation and initial conditions, $y(t_0) = y_0, y'(t_0) = y_1, ..., y^{(n-1)}(t_0) = y_{n-1}$. IVP