Derivation of general solutions:

If $b^2 - 4ac > 0$ we guessed e^{rt} is a solution and noted that any linear combination of solutions is a solution to a homogeneous linear differential equation.

Section 3.3: If $b^2 - 4ac < 0$, :

Changed format of $y = c_1e^{r_1t} + c_2e^{r_2t}$ to linear combination of real-valued functions instead of complex valued functions by using Euler's formula:

$$e^{it} = cos(t) + isin(t)$$

Hence $e^{(d+in)t} = e^{dt}e^{int} = e^{dt}[cos(nt) + isin(nt)]$

Let $r_1 = d + in$, $r_2 = d - in$

$$y = c_1e^{r_1t} + c_2e^{r_2t}$$
$$= c_1e^{dt}[cos(nt)+isin(nt)]+c_2e^{dt}[cos(-nt)+isin(-nt)]$$
$$= c_1e^{dt}cos(nt)+ic_1e^{dt}sin(nt)+c_2e^{dt}cos(nt)-ic_2e^{dt}sin(nt)$$
$$=(c_1 + c_2)e^{dt}cos(nt) + i(c_1 - c_2)e^{dt}sin(nt)$$
$$= k_1e^{dt}cos(nt) + k_2e^{dt}sin(nt)$$

---

repeated root

Section 3.4: If $b^2 - 4ac = 0$, then $r_1 = r_2$.
Hence one solution is $y = e^{r_1t}$ Need second solution.

If $y = e^{rt}$ is a solution, $y = ce^{rt}$ is a solution.

How about $y = v(t)e^{rt}$?

$$y' = v'(t)e^{rt} + v(t)re^{rt}$$
$$y'' = v''(t)e^{rt} + v'(t)re^{rt} + v'(t)re^{rt} + v(t)r^2e^{rt}$$
$$= v''(t)e^{rt} + 2v'(t)re^{rt} + v(t)r^2e^{rt}$$

$$ay'' + by' + cy = 0$$

$$a(v''e^{rt}+2v're^{rt}+vr^2e^{rt})+b(v'e^{rt}+vre^{rt})+cve^{rt} = 0$$

$$a(v''(t)+2v'(t)r+v(t)r^2)+b(v'(t)+v(t)r)+cv(t) = 0$$

$$av''(t)+2av'(t)r+av(t)r^2+bv'(t)+bv(t)r+cv(t) = 0$$

$$av''(t) + (2ar + b)v'(t) + (ar^2 + br + c)v(t) = 0$$

$$av''(t) + (2a(\tfrac{-b}{2a}) + b)v'(t) + 0 = 0$$

since $ar^2 + br + c = 0$ and $r = \frac{-b}{2a}$

$av''(t) + (-b + b)v'(t) = 0$. Thus $av''(t) = 0$.

Hence $v''(t) = 0$ and $v'(t) = k_1$ and $v(t) = k_1t + k_2$

Hence $v(t)e^{r_1t} = (k_1t + k_2)e^{r_1t}$ is a soln

Thus te^{r_1t} is a nice second solution.

Hence general solution is $y = c_1e^{r_1t} + c_2te^{r_1t}$

3.6

Solve $y'' - 2y' + y = e^t ln(t)$

1) Find homogeneous solutions: Solve $y'' - 2y' + y = 0$

Guess: $y = e^{rt}$, then $y' = re^{rt}$, $y'' = r^2e^{rt}$, and

$r^2e^{rt} - 2re^{rt} + e^{rt} = 0$ implies $\boxed{r^2 - 2r + 1 = 0}$

$(r-1)^2 = 0$, and hence $r = 1$

General homogeneous solution: $y = c_1e^t + c_2te^t$

since have two linearly independent solutions: $\{e^t, te^t\}$

2.) Find a non-homogeneous solution:

- Sect. 3.5 method: Educated guess ← undetermined coeff

variation of parameters

Sect. 3.6: **Guess** $\boxed{y = u_1(t)e^t + u_2(t)te^t}$ **and solve for u_1 and u_2**

$$u_1(t) = -\int \frac{\phi_2(t)g(t)}{W(\phi_1,\phi_2)}dt = -\int \frac{(te^t)(e^t ln(t))}{e^{2t}}dt = -\int t ln(t)$$
$$= -\left[\frac{t^2 ln(t)}{2} - \int \frac{t}{2}\right] = -\frac{t^2 ln(t)}{2} + \frac{t^2}{4}$$

$$u_2(t) = \int \frac{\phi_1(t)g(t)}{W(\phi_1,\phi_2)}dt = \int \frac{(e^t)(e^t ln(t))}{e^{2t}}dt = \int ln(t) = t ln(t) - t$$

$$W(\phi_1,\phi_2) = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t} + te^{2t} - te^{2t}$$

$u = ln(t)$ $\quad dv = tdt$
$du = \frac{dt}{t}$ $\quad v = \frac{t^2}{2}$

$u = ln(t)$ $\quad dv = dt$
$du = \frac{dt}{t}$ $\quad v = t$

General solution : $y = c_1e^t + c_2te^t + \boxed{(-\frac{t^2 ln(t)}{2} + \frac{t^2}{4})e^t + (t ln(t) - t)te^t}$

$u_1 \quad \phi_1 \quad u_2 \quad \phi_2$

which simplifies to $y = c_1e^t + c_2te^t + (\frac{ln(t)}{2} - \frac{3}{4})t^2e^t$

$$y = \underbrace{c_1\phi_1 + c_2\phi_2}_{\text{homo}} + \underbrace{u_1\phi_1 + u_2\phi_2}_{\text{nonho}}$$

FYI

Sect.3.6: **Guess** $\boxed{y = u_1(t)e^t + u_2(t)te^t}$ **and solve for u_1 and u_2**

$y' = u_1'e^t + u_1e^t + u_2'te^t + u_2(e^t + te^t) \cancel{= e^{2t} + te^{2t} - te^{2t} - e^{2t}}$

Two unknown functions, u_1 and u_2, but only one equation ($y'' - 2y' + y = e^t ln(t)$). Thus might be OK to choose 2nd eq'n.

Avoid 2nd derivative in y'': $\boxed{\textbf{Choose } u_1'e^t + u_2'te^t = 0}$ ← 2nd eqn

Hence $y' = u_1e^t + u_2(e^t + te^t)$.

and $y'' = u_1'e^t + u_1e^t + u_2'(e^t + te^t) + u_2(e^t + e^t + te^t)$.
$= u_1'e^t + u_1e^t + u_2'e^t + u_2'te^t + u_2(2e^t + te^t)$.
$= u_1e^t + u_2'e^t + u_2(2e^t + te^t)$.

Solve $y'' - 2y' + y = e^t ln(t)$ 1 eqn

$u_1e^t + u_2'e^t + u_2(2e^t + te^t) - 2[u_1e^t + u_2(e^t + te^t)] + u_1e^t + u_2te^t = e^t ln(t)$

$u_2'e^t + \cancel{2u_2e^t} + \cancel{u_2te^t} - \cancel{2u_2e^t} - 2u_2te^t + \cancel{u_2te^t} = e^t ln(t)$

$u_2' = ln(t)$ or in other words, $\frac{du_2}{dt} = ln(t)$

Thus $\int du_2 = \int ln(t)dt$

$u_2 = t ln(t) - t$. Note only need one solution, so don't need $+C$.

$y = u_1(t)e^t + [t ln(t) - t]te^t$

$u_1'e^t + u_2'te^t = 0$. Thus $u_1' + u_2't = 0$. Hence $u_1' = -u_2't = -t ln(t)$

Thus $u_1 = -\int t ln(t)dt = -\frac{t^2 ln(t)}{2} + \frac{t^2}{4}$

Thus the general solution is

$$y = c_1e^t + c_2te^t + (-\frac{t^2 ln(t)}{2} + \frac{t^2}{4})e^t + (t ln(t) - t)te^t$$

$\uparrow kL$ 3.7

$\downarrow mg$

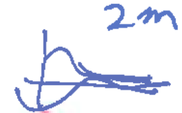
$$\boxed{\begin{gathered} mg - kL = 0 \\ mu''(t) + \gamma u'(t) + ku(t) = 0, \quad m, \gamma, k \geq 0 \\ m, k > 0 \end{gathered}}$$

$$r_1, r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m} \Longleftarrow mr^2 + \gamma r + k = 0$$

$\gamma^2 - 4km > 0$: $u(t) = Ae^{r_1 t} + Be^{r_2 t}$

$\gamma^2 - 4km = 0$: $u(t) = (A + Bt)e^{r_1 t}$, $r_1 = -\frac{\gamma}{2m}$

As $t \to \infty$, $u(t) \to 0$



$\gamma^2 - 4km < 0$: $u(t) = e^{-\frac{\gamma t}{2m}}(Acos\mu t + Bsin\mu t)$
$= e^{-\frac{\gamma t}{2m}}(Rcos(\mu t - \delta))$
where $A = Rcos(\delta)$, $B = Rsin(\delta)$

μ = quasi frequency, $\frac{2\pi}{\mu}$ = quasi period

$$r_1, r_2 = \frac{-\gamma \pm i\sqrt{4km - \gamma^2}}{2m}$$

$$y = c_1 e^{-\frac{\gamma}{2m}t} \cos\left(\frac{t\sqrt{4km-\gamma^2}}{2m}\right) + c_2 e^{-\frac{\gamma t}{2m}} \sin\left(\frac{t\sqrt{4km-\gamma^2}}{2m}\right)$$

Note if $\gamma = 0$, then

No damping, no negative exponential term, so oscillates forever

Critical damping: $\gamma = 2\sqrt{km}$

Overdamped: $\gamma > 2\sqrt{km}$

Suppose a mass weighs 64 lbs stretches a spring 4 ft. If there is no damping and the spring is stretched an additional foot and set in motion with an upward velocity of $\sqrt{8}$ ft/sec, find the equation of motion of the mass.

$Weight = mg$: $m = \frac{weight}{g} = \frac{64}{32} = 2$

$mg - kL = 0$ implies $k = \frac{mg}{L} = \frac{64}{4} = 16$

$mu''(t) + \gamma u'(t) + ku(t) = F_{external}$

[$\gamma^2 - 4km < 0$: $u(t) = e^{-\frac{\gamma t}{2m}}(Acos\mu t + Bsin\mu t)$
Hence $u(t) = Acos\mu t + Bsin\mu t$ since $\gamma = 0$].

$2u''(t) + 16u(t) = 0$

$u''(t) + 8u(t) = 0, \qquad u(0) = 1,\ u'(0) = -\sqrt{8}$

$r^2 + 8 = 0 \to r = \pm\sqrt{-8} = \pm i\sqrt{8} = 0 \pm i\sqrt{8}$

$u(t) = c_1 e^{it\sqrt{8}} + c_2 e^{-it\sqrt{8}}$

$u(t) = Acos\sqrt{8}t + Bsin\sqrt{8}t$

$u(0) = 1$: $1 = Acos(0) + Bsin(0) = A$

$u'(t) = -\sqrt{8}Asin\sqrt{8}t + \sqrt{8}Bcos\sqrt{8}t$

$u'(0) = -\sqrt{8}$: $-\sqrt{8} = -\sqrt{8}Asin(0) + \sqrt{8}Bcos(0)$

$B = -1$ Thus $u(t) = cos\sqrt{8}t - sin\sqrt{8}t$