Solve $y'' - 4y' - 5y = 4sin(3t)$, $y(0) = 6$, $y'(0) = 7$.

1.) Find the general solution to $y'' - 4y' - 5y = 0$:

Guess $y = e^{rt}$ for HOMOGENEOUS equation:

$y' = re^{rt}$, $y' = r^2e^{rt}$

$y'' - 4y' - 5y = 0$

$r^2e^{rt} - 4re^{rt} - 5e^{rt} = 0$

$e^{rt}(r^2 - 4r - 5) = 0$

$e^{rt} \neq 0$, thus can divide both sides by e^{rt}:

$$r^2 - 4r - 5 = 0$$

$(r+1)(r-5) = 0$. Thus $r = -1, 5$.

Thus $y = e^{-t}$ and $y = e^{5t}$ are both solutions to HOMOGENEOUS equation.

Thus the general solution to the 2nd order linear HOMOGENEOUS equation is

$$y = c_1e^{-t} + c_2e^{5t}$$

2.) Find a solution to $ay'' + by' + cy = 4sin(3t)$:

Guess $y = Asin(3t) + Bcos(3t)$ ← 3.5 Method of undetermined coeff

$y' = 3Acos(3t) - 3Bsin(3t)$

$y'' = -9Asin(3t) - 9Bcos(3t)$

$y'' - 4y' - 5y = 4sin(3t)$

$$\begin{array}{lrcrcl}
y'' & -9Asin(3t) & - & 9Bcos(3t) & & \\
-4y' & 12Bsin(3t) & - & 12Acos(3t) & & \\
-5y & -5Asin(3t) & - & 5Bcos(3t)\,. & & \\
\hline
 & (12B - 14A)sin(3t) & + & (-14B - 12A)cos(3t) & = & 4sin(3t)
\end{array}$$

helps to keep organized

Since $\{sin(3t), cos(3t)\}$ is a linearly independent set:

$12B - 14A = 4$ and $-14B - 12A = 0$

Thus $A = -\frac{14}{12}B = -\frac{7}{6}B$ and

$12B - 14(-\frac{7}{6}B) = 12B + 7(\frac{7}{3}B) = \frac{36+49}{3}B = \frac{85}{3}B = 4$

Thus $B = 4(\frac{3}{85}) = \frac{12}{85}$ and $A = -\frac{7}{6}B = -\frac{7}{6}(\frac{12}{85}) = -\frac{14}{85}$

Thus $y = (-\frac{14}{85})sin(3t) + \frac{12}{85}cos(3t)$ is one solution to the non-homogeneous equation.

Thus the general solution to the 2nd order linear nonhomogeneous equation is

$$y = \underbrace{c_1e^{-t} + c_2e^{5t}}_{\text{general homog}} \underbrace{- (\frac{14}{85})sin(3t) + \frac{12}{85}cos(3t)}_{\text{a nonhom}}$$

3.7: Mechanical and Electrical Vibrations

Trig background:

$cos(y \mp x) = cos(x \mp y) = cos(x)cos(y) \pm sin(x)sin(y)$

Let $A = Rcos(\delta)$, $B = Rsin(\delta)$ in

$Acos(\omega_0 t) + Bsin(\omega_0 t)$
$= Rcos(\delta)cos(\omega_0 t) + Rsin(\delta)sin(\omega_0 t)$
$= Rcos(\omega_0 t - \delta)$

Amplitude $= R$
frequency $= \omega_0$ (measured in radians per unit time).
period $= \frac{2\pi}{\omega_0}$
phase (displacement) $= \delta$

3.7

and memorize and understand

Mechanical Vibrations:
$mu''(t) + \gamma u'(t) + ku(t) = F_{external}, \quad m, \gamma, k \geq 0$
$mg - kL = 0, \quad F_{damping}(t) = -\gamma u'(t)$
m= mass,
k = spring force proportionality constant,
γ = damping force proportionality constant
g = 9.8 m/sec^2 or $g = 32$ ft/sec^2

FYI

Electrical Vibrations:
$L\frac{dI(t)}{dt} + RI(t) + \frac{1}{C}Q(t) = E(t), \quad L, R, C \geq 0$ and $I = \frac{dQ}{dt}$

$lQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = E(t)$

L = inductance (henrys),
R = resistance (ohms)
C = capacitance (farads)
$Q(t)$ = charge at time t (coulombs)
$I(t)$ = current at time t (amperes)
$E(t)$ = impressed voltage (volts).

1 volt = 1 ohm · 1 ampere = 1 coulomb / 1 farad = 1 henry · 1 amperes/ 1 second

$$mu''(t) + \gamma u'(t) + ku(t) = 0, \quad m, \gamma, k \geq 0$$

$$r_1, r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}$$

$\gamma^2 - 4km > 0$: $u(t) = Ae^{r_1 t} + Be^{r_2 t}$

$\gamma^2 - 4km = 0$: $u(t) = (A + Bt)e^{r_1 t}$

$\gamma^2 - 4km < 0$: $u(t) = e^{-\frac{\gamma t}{2m}}(Acos\mu t + Bsin\mu t)$
$= e^{-\frac{\gamma t}{2m}} Rcos(\mu t - \delta)$
where $A = Rcos(\delta)$, $B = Rsin(\delta)$

μ = quasi frequency, $\frac{2\pi}{\mu}$ = quasi period

Note if $\gamma = 0$, then

Critical damping: $\gamma = 2\sqrt{km}$

Overdamped: $\gamma > 2\sqrt{km}$

use L to find k

$\gamma = 0$

Suppose a mass weighs 64 lbs stretches a spring 4 ft. If there is no damping and the spring is stretched an additional foot and set in motion with an upward velocity of $\sqrt{8}$ ft/sec, find the equation of motion of the mass.

negative direction

$Weight = mg$: $m = \frac{weight}{g} = \frac{64}{32} = 2$

$mg - kL = 0$ implies $k = \frac{mg}{L} = \frac{64}{4} = 16$

$mu''(t) + \gamma u'(t) + ku(t) = F_{external}$

[$\gamma^2 - 4km < 0$: $u(t) = e^{-\frac{\gamma t}{2m}}(Acos\mu t + Bsin\mu t)$
Hence $u(t) = Acos\mu t + Bsin\mu t$ since $\gamma = 0$].

$2u''(t) + 16u(t) = 0$

$u''(t) + 8u(t) = 0, \qquad u(0) = 1, \; u'(0) = -\sqrt{8}$

$r^2 + 8 = 0 \to r = \pm\sqrt{-8} = \pm i\sqrt{8} = 0 \pm i\sqrt{8}$

~~$u(t) = c_1 e^{it\sqrt{8}} + c_2 e^{-it\sqrt{8}}$~~ ugly unacceptable form

$\boxed{u(t) = Acos\sqrt{8}t + Bsin\sqrt{8}t}$

$u(0) = 1$: $1 = Acos(0) + Bsin(0) = A$

$u'(t) = -\sqrt{8}Asin\sqrt{8}t + \sqrt{8}Bcos\sqrt{8}t$

$u'(0) = -\sqrt{8}$: $-\sqrt{8} = -\sqrt{8}Asin(0) + \sqrt{8}Bcos(0)$

$B = -1 \qquad$ Thus $u(t) = cos\sqrt{8}t - sin\sqrt{8}t$