Solving first order differential equation:

Method 1 (sect. 2.2): Separate variables.

Method 2 (sect. 2.1): If linear $[y'(t)+p(t)y(t) = g(t)]$, multiply equation by an integrating factor $u(t) = e^{\int p(t)dt}$.

$$\begin{aligned} y' + py &= g \\ y'u + upy &= ug \\ (uy)' &= ug \\ \int (uy)' &= \int ug \\ uy &= \int ug \\ &\text{etc...} \end{aligned}$$

Method 3 (sect. 2.4): Solve Bernoulli's equation,

$$y' + p(t)y = g(t)y^n$$

$(y'y^{-n}) + p(t)(y^{1-n}) = g(t)$

when $n > 1$ by changing it to a linear equation by substituting $v = y^{1-n}$ $\Rightarrow v' = (1-n)(y^{-n}y')$

If $v = \frac{dx}{dt}$, can use the following to simplify (especially if there are 3 variables).

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

integration techniques: u-substitution, integration by parts, partial fractions.

---

direction field = slope field = graph of $\frac{dv}{dt}$ in t, v-plane.

*** can use slope field to determine behavior of v including as $t \to \infty$.

Equilibrium Solution = constant solution

stable, unstable, semi-stable.

Solving second order differential equation:

p. 135: $y'' = f(t, y')$, $y'' = f(y, y')$,

Transform to first order: Let $v = y'$.

If needed, note $v' = \frac{dv}{dt} = \frac{dv}{dt}\frac{dy}{dy} = \frac{dv}{dy}\frac{dy}{dt} = \frac{dv}{dy}v$.

Note this trick sometimes helpful for first order equations.

ch 2

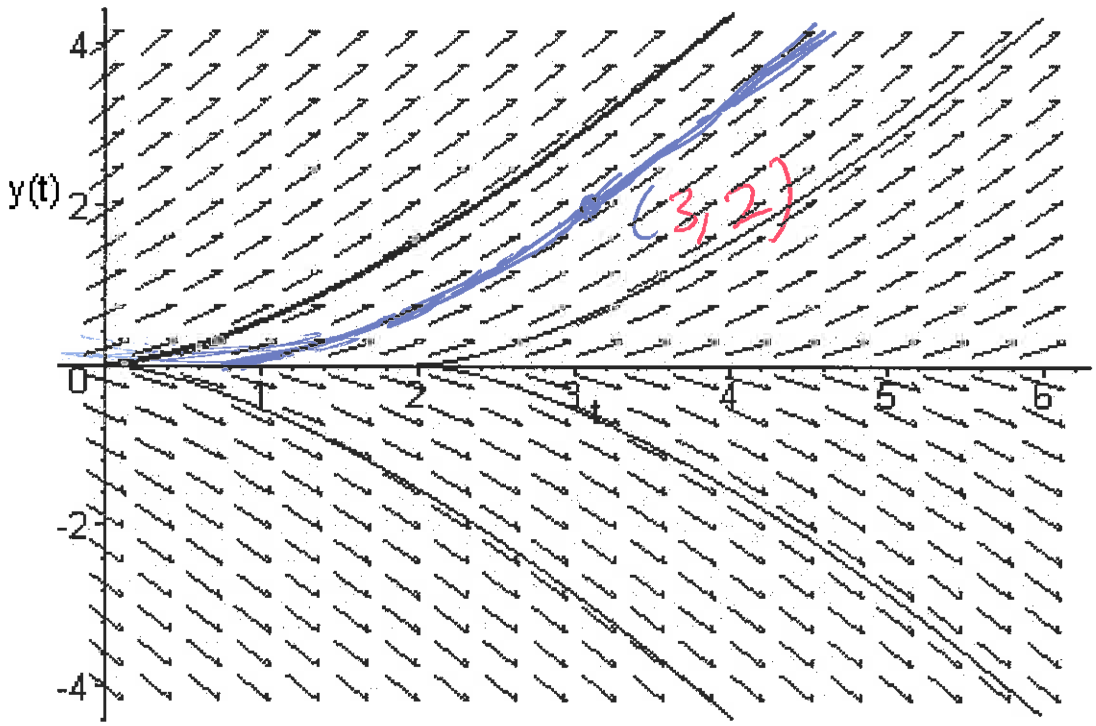
Ch 3: linear $ay'' + by' + cy = 0$, ← ch 3

Need to have two independent solutions.

$ar^2 + br + c = 0$

If ϕ_1, ϕ_2 are solutions to a LINEAR HOMOGENEOUS differential equation, $c_1\phi_1 + c_2\phi_2$ is also a solution

$y' = y^{1/3}$



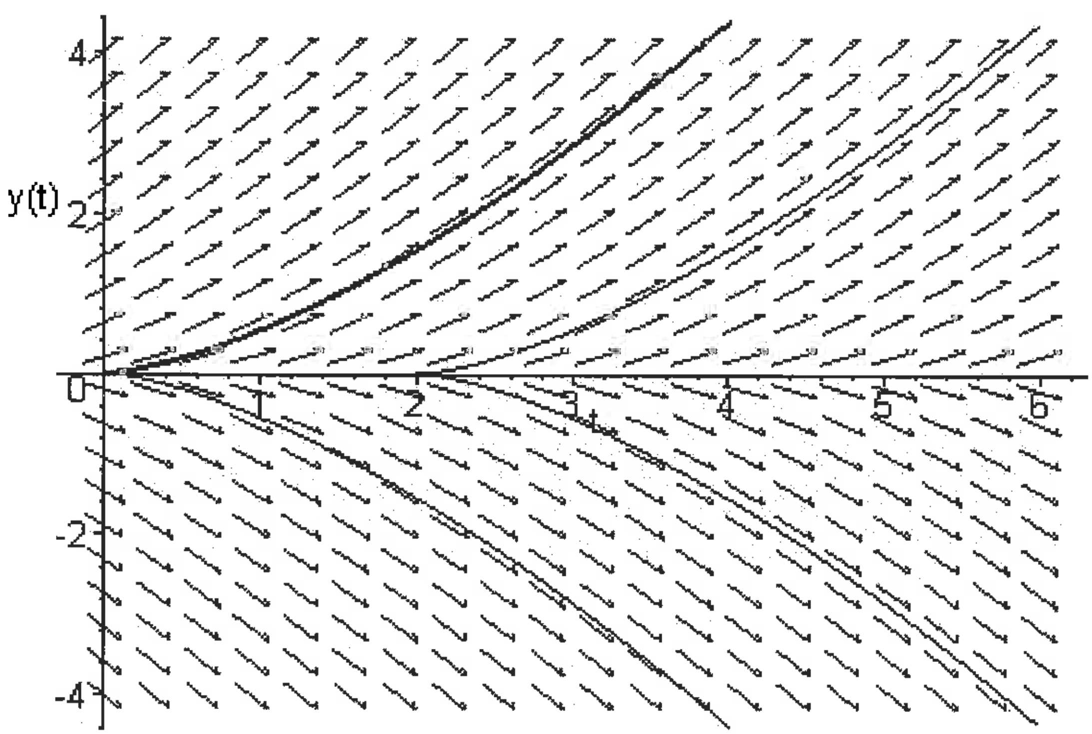


Figure 2.4.1 from ***Elementary Differential Equations and Boundary Value Problems*, Eighth Edition** by William E. Boyce and Richard C. DiPrima

Existence and Uniqueness

1st order LINEAR differential equation:

Thm 2.4.1: If $p : (a,b) \to R$ and $g : (a,b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a,b) \to R$ that satisfies the initial value problem

$$y' + p(t)y = g(t),$$
$$y(t_0) = y_0$$

2nd order LINEAR differential equation:

Thm 3.2.1: If $p : (a,b) \to R$, $q : (a,b) \to R$, and $g : (a,b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a,b) \to R$ that satisfies the initial value problem

$$y'' + p(t)y' + q(t)y = g(t),$$
$$y(t_0) = y_0, \quad y'(t_0) = y_0'$$

Definition: The Wronskian of two differential functions, f and g is

$$W(f,g) = fg' - f'g = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$$

Thm 3.2.4: Given (1) the hypothesis of thm 3.2.1
(2) ϕ_1 and ϕ_2 are 2 sol'ns to $y'' + p(t)y' + q(t)y = 0$ (*)
(3) $W(\phi_1, \phi_2)(t_0) \neq 0$, for some $t_0 \in (a,b)$,
then if f is a solution to (*), then $f = c_1\phi_1 + c_2\phi_2$ for some c_1 and c_2.

Thm 2.4.2: Suppose $z = f(t,y)$ and $z = \frac{\partial f}{\partial y}(t,y)$ are continuous on $(a,b) \times (c,d)$ and the point $(t_0, y_0) \in (a,b) \times (c,d)$, then there exists an interval $(t_0 - h, t_0 + h) \subset (a,b)$ such that there exists a unique function $y = \phi(t)$ defined on $(t_0 - h, t_0 + h)$ that satisfies the following initial value problem:

$$y' = f(t,y), \quad y(t_0) = y_0.$$

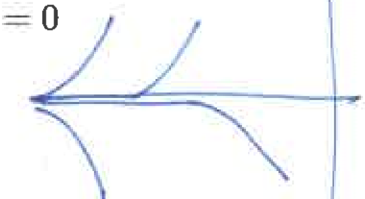
Note the initial value problem

$$y' = y^{\frac{1}{3}}, \; y(0) = 0$$

has an infinite number of different solutions.

$$y^{-\frac{1}{3}} dy = dt$$
$$\tfrac{3}{2} y^{\frac{2}{3}} = t + C$$
$$y = \pm(\tfrac{2}{3}t + C)^{\frac{3}{2}}$$
$$y(0) = 0 \text{ implies } C = 0$$

Thus $y = \pm(\frac{2}{3}t)^{\frac{3}{2}}$ are solutions.



$y = 0$ is also a solution, etc.

Compare to Thm 2.4.2:
$f(t,y) = y^{\frac{1}{3}}$ is continuous near (0, 0)
But $\frac{\partial f}{\partial y}(t,y) = \frac{1}{3}y^{\frac{-2}{3}}$ is not continuous near (0, 0) since it isn't defined at (0, 0).

3.) If initial value problem:

Once general solution is known, can solve initial value problem (i.e., use initial conditions to find c_1, c_2).

NOTE: you must know the GENERAL solution to the ODE BEFORE you can solve for the initial values. The homogeneous solution and the one nonhomogeneous solution found in steps 1 and 2 above do NOT need to satisfy the initial values.

Solve $y'' - 4y' - 5y = 4sin(3t)$, $y(0) = 6$, $y'(0) = 7$.

General solution: $y = c_1e^{-t} + c_2e^{5t} - (\frac{14}{85})sin(3t) + \frac{12}{85}cos(3t)$

Thus $y' = -c_1e^{-t} + 5c_2e^{5t} - (\frac{42}{85})cos(3t) - \frac{36}{85}sin(3t)$

$y(0) = 6$: $\quad 6 = c_1 + c_2 + \frac{12}{85} \qquad \frac{498}{85} = c_1 + c_2$

$y'(0) = 7$: $\quad 7 = -c_1 + 5c_2 - \frac{42}{85} \qquad \frac{637}{85} = -c_1 + 5c_2$

$6c_2 = \frac{498+637}{85} = \frac{1135}{85} = \frac{227}{17}$. Thus $c_2 = \frac{227}{102}$.

$c_1 = \frac{498}{85} - c_2 = \frac{498}{85} - \frac{227}{102} = \frac{2988-1135}{510} = \frac{1853}{510} = \frac{109}{30}$

Thus $y = (\frac{109}{30})e^{-t} + (\frac{227}{102})e^{5t} - (\frac{14}{85})sin(3t) + \frac{12}{85}cos(3t)$.

Partial Check: $y(0) = (\frac{109}{30}) + (\frac{227}{102}) + \frac{12}{85} = 6$.

$y'(0) = -\frac{109}{30} + 5(\frac{227}{102}) - \frac{42}{85} = 7$.

$\left[\left(\frac{2}{3}t\right)^{3/2}\right]' = \frac{3}{2}\left(\frac{2}{3}t\right)^{1/2} \cdot \left(\frac{2}{3}\right) = \frac{2}{3}t^{1/2}$

Potential proofs for exam 1:

Proof by (counter) example:

1. Prove a function is not 1:1, not onto, not a bijection, not linear.

see $y' = y^{1/3}$ ex.

2. Prove that a differential equation can have multiple solutions.

Prove convergence of a series using ratio test.

Induction proof.

Prove a function is linear.

Theorem 3.2.2: If $y = \phi_1(t)$ and $y = \phi_2(t)$ are solutions to the 2nd order linear ODE, $ay'' + by' + cy = 0$, then their linear combination $y = c_1\phi_1(t) + c_2\phi_2(t)$ is also a solution for constants c_1 and c_2.

Note you may use what you know from both pre-calculus and calculus (e.g., integration and derivatives are linear).

$y = 0$ is a soln to $y' = y^{1/3}$

$y(0) = 0$

Pf: $y = 0 \Rightarrow y' = 0 \qquad 0 = 0^{1/3}$ ✓

$\Rightarrow y(0) = 0$ ✓

$y = \left(\frac{2}{3}t\right)^{3/2} \Rightarrow y(0) = 0$

~~$y' = \frac{2}{3}t^{1/2}$~~

$(y^{1/3}) = \left(\left(\frac{2}{3}t\right)^{3/2}\right)^{1/3}$

$\boxed{2.8}$

$\boxed{\text{Given: } \boxed{y' = f(t,y),\ y(0) = 0} \qquad \text{Eqn (*)}}$

$f, \partial f/\partial y$ continuous $\forall (t,y) \in (-a,a) \times (-b,b)$. Then

$y = \phi(t)$ is a solution to (*) iff

$\phi'(t) = f(t, \phi(t)), \quad \phi(0) = 0$ iff

$\int_0^t \phi'(s)ds = \int_0^t f(s, \phi(s))ds, \quad \phi(0) = 0$ iff

$\phi(t) = \phi(t) - \phi(0) = \int_0^t f(s, \phi(s))ds$

Thus $y = \phi(t)$ is a solution to (*) iff $\boxed{\phi(t) = \int_0^t f(s, \phi(s))ds}$

Construct ϕ using method of successive approximation – also called Picard's iteration method.

Let $\phi_0(t) = 0$ (or the function of your choice)

Let $\phi_1(t) = \int_0^t f(s, \phi_0(s))ds$

Let $\phi_2(t) = \int_0^t f(s, \phi_1(s))ds$

⋮

Let $\boxed{\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))ds}$

Let $\phi(t) = \lim_{n\to\infty} \phi_n(t)$

week 15

For specific case:

Some questions:

1.) Does $\phi_n(t)$ exist for all n? Pf by induction

2.) Does sequence ϕ_n converge? Examples Ratio test

3.) Is $\phi(t) = \lim_{n\to\infty} \phi_n(t)$ a solution to (*)? Plug it in

4.) Is the solution unique. ⇐ week 15

Example: $y' = t + 2y$. That is $f(t,y) = t + 2y$, $\frac{\partial f}{\partial y} = 2$ } cont $\forall (t,y)$

Let $\phi_0(t) = 0$

Let $\phi_1(t) = \int_0^t f(s,0)ds = \int_0^t (s + 2(0))ds$

$$= \int_0^t s\,ds = \frac{s^2}{2}\Big|_0^t = \frac{t^2}{2}$$

Let $\phi_2(t) = \int_0^t f(s, \phi_1(s))ds = \int_0^t f(s, \frac{s^2}{2})ds$

$$= \int_0^t (s + 2(\frac{s^2}{2}))ds = \frac{t^2}{2} + \frac{t^3}{3}$$

Let $\phi_3(t) = \int_0^t f(s, \phi_2(s))ds = \int_0^t f(s, \frac{s^2}{2} + \frac{s^3}{3})ds$

$$= \int_0^t (s + 2\frac{s^2}{2} + \frac{s^3}{3}))ds = \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6}$$

⋮

See class notes.

$$\phi_n(t) = \sum_{i=1}^{n} \frac{2^{i-1} t^{i+1}}{(i+1)!}$$

Solve $y'' - 4y' - 5y = 4sin(3t)$, $y(0) = 6$, $y'(0) = 7$.

1.) Find the general solution to $y'' - 4y' - 5y = 0$:

Guess $y = e^{rt}$ for HOMOGENEOUS equation:

$y' = re^{rt}$, $y' = r^2e^{rt}$

$y'' - 4y' - 5y = 0$

$r^2e^{rt} - 4re^{rt} - 5e^{rt} = 0$

$e^{rt}(r^2 - 4r - 5) = 0$

$e^{rt} \neq 0$, thus can divide both sides by e^{rt}:

$$r^2 - 4r - 5 = 0$$

$\boxed{(r+1)(r-5) = 0. \text{ Thus } r = -1, 5.}$

Thus $y = e^{-t}$ and $y = e^{5t}$ are both solutions to HOMOGENEOUS equation.

Thus the general solution to the 2nd order linear HOMOGENEOUS equation is

$$\boxed{y = c_1e^{-t} + c_2e^{5t}}$$

2.) Find a solution to $ay'' + by' + cy = 4sin(3t)$:

nonhomog

Guess $y = Asin(3t) + Bcos(3t)$

2 unknowns need 2 eqns

$y' = 3Acos(3t) - 3Bsin(3t)$

$y'' = -9Asin(3t) - 9Bcos(3t)$

$y'' - 4y' - 5y = 4sin(3t)$

$$\begin{array}{rcl} -9Asin(3t) & - & 9Bcos(3t) \quad y'' \\ 12Bsin(3t) & - & 12Acos(3t) \quad -4y' \\ -5Asin(3t) & - & 5cos(3t) \quad -5y \\ \hline (12B - 14A)sin(3t) & + & (-14B - 12A)cos(3t) = 4sin(3t) + 0cos(3t) \end{array}$$

Since $\{sin(3t), cos(3t)\}$ is a linearly independent set:

$\boxed{12B - 14A = 4 \text{ and } -14B - 12A = 0}$

Thus $A = -\frac{14}{12}B = -\frac{7}{6}B$ and

$12B - 14(-\frac{7}{6}B) = 12B + 7(\frac{7}{3}B) = \frac{36+49}{3}B = \frac{85}{3}B = 4$

Thus $B = 4(\frac{3}{85}) = \frac{12}{85}$ and $A = -\frac{7}{6}B = -\frac{7}{6}(\frac{12}{85}) = -\frac{14}{85}$

Thus $y = (-\frac{14}{85})sin(3t) + \frac{12}{85}cos(3t)$ is one solution to the non-homogeneous equation.

Thus the general solution to the 2nd order linear nonhomogeneous equation is

$$y = \underbrace{c_1e^{-t} + c_2e^{5t}}_{\text{homog}} \underbrace{- (\frac{14}{85})sin(3t) + \frac{12}{85}cos(3t)}_{+\text{ a nonhomog soln}}$$

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$c_1 = \frac{498}{85} - c_2 = \frac{498}{85} - \frac{227}{102} = \frac{2988-1135}{510} = \frac{1853}{510} = \frac{109}{30}$

Thus $\boxed{y = (\frac{109}{30})e^{-t} + (\frac{227}{102})e^{5t} - (\frac{14}{85})sin(3t) + \frac{12}{85}cos(3t)}$. Soln to IVP

Answer

Partial Check: $y(0) = (\frac{109}{30}) + (\frac{227}{102}) + \frac{12}{85} = 6$.

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