Ch 3

Derivation of general solutions:

3.1 If $b^2 - 4ac > 0$ we guessed e^{rt} is a solution and noted that any linear combination of solutions is a solution to a homogeneous linear differential equation.

Section 3.3: If $b^2 - 4ac < 0$, :

Changed format of $y = c_1e^{r_1t} + c_2e^{r_2t}$ to linear combination of real-valued functions instead of complex valued functions by using Euler's formula: ugly when r_1, r_2 are complex

$$\boxed{e^{it} = cos(t) + isin(t)}$$

Hence $e^{(d+in)t} = e^{dt}e^{int} = e^{dt}[cos(nt) + isin(nt)]$

Let $r_1 = d + in$, $r_2 = d - in$

$y = c_1e^{r_1t} + c_2e^{r_2t}$ ⇐ ugly

$= c_1e^{dt}[cos(nt)+isin(nt)]+c_2e^{dt}[cos(-nt)+isin(-nt)]$

$= c_1e^{dt}cos(nt)+ic_1e^{dt}sin(nt)+c_2e^{dt}cos(nt)-ic_2e^{dt}sin(nt)$

$=(c_1 + c_2)e^{dt}cos(nt) + i(c_1 - c_2)e^{dt}sin(nt)$

$y = k_1e^{dt}cos(nt) + k_2e^{dt}sin(nt)$ ⇐ acceptable

Section 3.4: If $b^2 - 4ac = 0$, then $r_1 = r_2$.
Hence one solution is $y = e^{r_1t}$ Need second solution.

If $y = e^{rt}$ is a solution, $y = ce^{rt}$ is a solution.

How about $\boxed{y = v(t)e^{rt}?}$ ⇐ Suppose this is a soln

$y' = v'(t)e^{rt} + v(t)re^{rt}$

$y'' = v''(t)e^{rt} + v'(t)re^{rt} + v'(t)re^{rt} + v(t)r^2e^{rt}$
$\quad = v''(t)e^{rt} + 2v'(t)re^{rt} + v(t)r^2e^{rt}$

$ay'' + by' + cy = 0$

$a(v''\cancel{e^{rt}}+2v'r\cancel{e^{rt}}+vr^2\cancel{e^{rt}})+b(v'\cancel{e^{rt}}+vr\cancel{e^{rt}})+cv\cancel{e^{rt}} = 0$

$a(v''(t)+2v'(t)r+v(t)r^2)+b(v'(t)+v(t)r)+cv(t) = 0$

$av''(t)+2av'(t)r+av(t)r^2+bv'(t)+bv(t)r+cv(t) = 0$

$av''(t) + (2ar + b)v'(t) + \underbrace{(ar^2 + br + c)}_{=0}v(t) = 0$ ⇐ e^{rt} soln!

$av''(t) + (2a(\boxed{\frac{-b}{2a}}) + b)v'(t) + 0 = 0$

since $ar^2 + br + c = 0$ and $\boxed{r = \frac{-b}{2a}}$

$av''(t) + \cancel{(-b + b)v'(t)}^{0} = 0.$ Thus $av''(t) = 0$.

Hence $v''(t) = 0$ and $v'(t) = k_1$ and $v(t) = k_1t + k_2$

Hence $v(t)e^{r_1t} = (k_1t + k_2)e^{r_1t}$ is a soln

Thus te^{r_1t} is a nice second solution.

Hence general solution is $\boxed{y = c_1e^{r_1t} + c_2te^{r_1t}}$ one repeated root

Guess $y = e^{rt} \to r^2 \cancel{e^{rt}} + \cancel{e^{rt}} = 0 \Rightarrow r^2 + 1 = 0$

Solve: $y'' + y = 0$, $y(0) = -1$, $y'(0) = -3$ ← IVP

$r^2 + 1 = 0$ implies $r^2 = -1$. Thus $r = \pm i$.

Since $r = 0 \pm 1i$, $\boxed{y = k_1 cos(t) + k_2 sin(t)}$. Then $y' = -k_1 sin(t) + k_2 cos(t)$

$y(0) = -1$: $-1 = k_1 cos(0) + k_2 sin(0)$ implies $-1 = k_1$

$y'(0) = -3$: $-3 = -k_1 sin(0) + k_2 cos(0)$ implies $-3 = k_2$

Thus IVP solution: $y = -cos(t) - 3sin(t)$

When does the following IVP have a unique solution:

3.2

IVP: $ay'' + by' + cy = 0$, $y(t_0) = y_0$, $y'(t_0) = y_1$.

Suppose $y = c_1\phi_1(t) + c_2\phi_2(t)$ is a solution to $ay'' + by' + cy = 0$. Then $y' = c_1\phi_1'(t) + c_2\phi_2'(t)$

$y(t_0) = y_0$: $y_0 = c_1\phi_1(t_0) + c_2\phi_2(t_0)$

$y'(t_0) = y_1$: $y_1 = c_1\phi_1'(t_0) + c_2\phi_2'(t_0)$

To find IVP solution, need to solve above system of two equations for the unknowns c_1 and c_2.

Note the IVP has a unique solution if and only if the above system of two equations has a unique solution for c_1 and c_2.

Note that in these equations c_1 and c_2 are the unknowns and $y_0, \phi_1(t_0), \phi_2(t_0), y_1, \phi_1'(t_0), \phi_2'(t_0)$ are the constants. We can translate this linear system of equations into matrix form:

$$\begin{array}{l} c_1\phi_1(t_0) + c_2\phi_2(t_0) = y_0 \\ c_1\phi_1'(t_0) + c_2\phi_2'(t_0) = y_1 \end{array} \quad \text{implies} \quad \begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$$

Note this equation has a unique solution if and only if $det \begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) \end{bmatrix} = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} = \phi_1\phi_2' - \phi_1'\phi_2 \neq 0$

Definition: The Wronskian of two differential functions, ϕ_1 and ϕ_2 is

$$\boxed{W(\phi_1, \phi_2) = \phi_1\phi_2' - \phi_1'\phi_2 = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}}$$

derivative

Examples:

1.) Wronskian of $cos(t), sin(t) = \begin{vmatrix} cos(t) & sin(t) \\ -sin(t) & cos(t) \end{vmatrix} = \boxed{cos^2(t) + sin^2(t) = 1} > 0.$

deriv

Thus $y = cos\,t$ & $y = sin\,t$ is a basis for soln set to $y'' + y = 0$

2.) Wronskian of $e^{dt}cos(nt)$, $e^{dt}sin(nt) = \begin{vmatrix} e^{dt}cos(nt) & e^{dt}sin(nt) \\ de^{dt}cos(nt) - ne^{dt}sin(nt) & de^{dt}sin(nt) + ne^{dt}cos(nt) \end{vmatrix}$

$= e^{dt}cos(nt)[de^{dt}sin(nt) + ne^{dt}cos(nt)] - e^{dt}sin(nt)[de^{dt}cos(nt) - ne^{dt}sin(nt)]$

$= e^{2dt}(cos(nt)[dsin(nt) + ncos(nt)] - sin(nt)[dcos(nt) - nsin(nt)])$

$= e^{2dt}(dcos(nt)sin(nt) + ncos^2(nt)] - dsin(nt)cos(nt) + nsin^2(nt)])$

d·t

$= e^{2dt}(ncos^2(nt) + nsin^2(nt)) = ne^{2dt}(cos^2(nt) + sin^2(nt)) = ne^{2dt} > 0$ for all t.

$r = d \pm in$, $d \in \mathbb{R}$, $n > 0$

Solve: $y'' + y = 0$, $y(0) = -1$, $y'(0) = -3$

$r^2 + 1 = 0$ implies $r^2 = -1$. Thus $r = \pm i$.

RECOMMENDED Method:

Since $r = 0 \pm 1i$, $y = k_1 cos(t) + k_2 sin(t)$

Then $y' = -k_1 sin(t) + k_2 cos(t)$

$y(0) = -1$: $-1 = k_1 cos(0) + k_2 sin(0)$ implies $-1 = k_1$

$y'(0) = -3$: $-3 = -k_1 sin(0) + k_2 cos(0)$ implies $-3 = k_2$

Thus IVP solution: $y = -cos(t) - 3sin(t)$

NOT RECOMMENDED: work with $y = c_1 e^{it} + c_2 e^{-it}$

unacceptably ugly

$y' = ic_1 e^{it} - ic_2 e^{-it}$

$y(0) = -1$: $-1 = c_1 e^0 + c_2 e^0$ implies $-1 = c_1 + c_2$.

$y'(0) = -3$: $-3 = ic_1 e^0 - ic_2 e^0$ implies $-3 = ic_1 - ic_2$.

$-1i = ic_1 + ic_2$.

$-3 = ic_1 - ic_2$.

$2ic_1 = -3 - i$ implies $c_1 = \frac{-3i - i^2}{-2} = \frac{3i-1}{2}$

$2ic_2 = 3 - i$ implies $c_2 = \frac{3i - i^2}{-2} = \frac{-3i-1}{2}$

Euler's formula: $e^{ix} = cos(x) + isin(x)$

$y = (\frac{3i-1}{2})e^{it} + (\frac{-3i-1}{2})e^{-it} = (\frac{3i-1}{2})[cos(t) + isin(t)] + (\frac{-3i-1}{2})[cos(-t) + isin(-t)]$

$= (\frac{3i-1}{2})[cos(t) + isin(t)] + (\frac{-3i-1}{2})[cos(t) - isin(t)]$

$= (\frac{3i}{2})cos(t) + (\frac{3i}{2})isin(t) + (\frac{-1}{2})cos(t) + (\frac{-1}{2})isin(t) + (\frac{-3i}{2})cos(t) - (\frac{-3i}{2})isin(t) + (\frac{-1}{2})cos(t) - (\frac{-1}{2})isin(t)$

$= (\frac{3i}{2})isin(t) + (\frac{-1}{2})cos(t) + (\frac{3i}{2})isin(t) + (\frac{-1}{2})cos(t)$

$= -(\frac{3}{2})sin(t) - (\frac{1}{2})cos(t) - (\frac{3}{2})sin(t) - (\frac{1}{2})cos(t)$

$= -3sin(t) - 1cos(t)$

pretty & acceptable answer

Need coef of 1 for y' for thm 2.4.1 AND also for solving when using integrating factor

Existence and Uniqueness

1st order LINEAR differential equation:

Thm 2.4.1: If $p : (a,b) \to R$ and $g : (a,b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\boxed{\phi : (a,b) \to R}$ that satisfies the initial value problem

$$1y' + p(t)y = g(t),$$
$$y(t_0) = y_0$$

Looked for max interval containing t_0 st p, g cont

2nd order LINEAR differential equation:

Thm 3.2.1: If $p : (a,b) \to R$, $q : (a,b) \to R$, and $g : (a,b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\boxed{\phi : (a,b) \to R}$ that satisfies the initial value problem

$$\boxed{1y'' + p(t)y' + q(t)y = g(t),}$$
$$y(t_0) = y_0, \quad y'(t_0) = y_0'$$

Look for max interval containing t_0 st p, q, g are cont

Definition: The Wronskian of two differential functions, f and g is

$$W(f,g) = fg' - f'g = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$$

Thm 3.2.4: Given (1) the hypothesis of thm 3.2.1
(2) ϕ_1 and ϕ_2 are 2 sol'ns to $y'' + p(t)y' + q(t)y = 0$ (*)
(3) $W(\phi_1, \phi_2)(t_0) \neq 0$, for some $t_0 \in (a,b)$,
then if f is a solution to (*), then $f = c_1\phi_1 + c_2\phi_2$ for some c_1 and c_2.

Need coeff of 1 for y'' for using thm 3.2.1 ONLY, don't need for solving

Thm 2.4.2: Suppose $z = f(t,y)$ and $z = \frac{\partial f}{\partial y}(t,y)$ are continuous on $(a,b) \times (c,d)$ and the point $(t_0, y_0) \in (a,b) \times (c,d)$, then there exists an interval $(t_0 - h, t_0 + h) \subset (a,b)$ such that there exists a unique function $y = \phi(t)$ defined on $(t_0 - h, t_0 + h)$ that satisfies the following initial value problem:

Non linear

$$y' = f(t,y), \quad y(t_0) = y_0.$$

Note the initial value problem

$$y' = y^{\frac{1}{3}}, \; y(0) = 0$$

$y(3) = 0$

has an infinite number of different solutions.

$$y^{-\frac{1}{3}} dy = dt$$
$$\tfrac{3}{2} y^{\frac{2}{3}} = t + C$$
$$\boxed{y = \pm(\tfrac{2}{3}t + C)^{\frac{3}{2}}}$$
$$y(0) = 0 \text{ implies } C = 0$$

Thus $y = \pm(\frac{2}{3}t)^{\frac{3}{2}}$ are solutions.

$\boxed{y = 0}$ is also a solution, etc.

Compare to Thm 2.4.2:
$f(t,y) = y^{\frac{1}{3}}$ is continuous near (0, 0)
But $\frac{\partial f}{\partial y}(t,y) = \frac{1}{3} y^{\frac{-2}{3}}$ is not continuous near (0, 0) since it isn't defined at (0, 0).