Second order differential equation:

Linear equation with constant coefficients:

If the second order differential equation is

$$ay'' + by' + cy = 0,$$

then $y = e^{rt}$ is a solution

Need to have two independent solutions.

Solve the following IVPs:

1.) $y'' - 6y' + 9y = 0$ $\qquad y(0) = 1,\ y'(0) = 2$

2.) $4y'' - y' + 2y = 0$ $\qquad y(0) = 3,\ y'(0) = 4$

3.) $4y'' + 4y' + y = 0$ $\qquad y(0) = 6,\ y'(0) = 7$

4.) $2y'' - 2y = 0$ $\qquad y(0) = 5,\ y'(0) = 9$

$ay'' + by' + cy = 0, \quad y = e^{rt}$, then
$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$ implies $ar^2 + br + c = 0$,

Suppose $r = r_1, r_2$ are solutions to $ar^2 + br + c = 0$

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $r_1 \neq r_2$, then $b^2 - 4ac \neq 0$. Hence a general solution is $y = c_1e^{r_1t} + c_2e^{r_2t}$

3,1

2 real soln

If $b^2 - 4ac > 0$, general solution is $y = c_1e^{r_1t} + c_2e^{r_2t}$.

3.3

If $b^2 - 4ac < 0$, change format to linear combination of real-valued functions instead of complex valued functions by using Euler's formula.

2 complex solns

general solution is $\boxed{y = c_1e^{dt}cos(nt) + c_2e^{dt}sin(nt)}$ where $r = d \pm in$

3,4

If $b^2 - 4ac = 0$, $r_1 = r_2$, so need 2nd (independent) solution: te^{r_1t}

1 real soln

Hence general solution is $\boxed{y = c_1e^{r_1t} + c_2te^{r_1t}}$.

Initial value problem: use $y(t_0) = y_0$, $y'(t_0) = y_0'$ to solve for c_1, c_2 to find unique solution.

Derivation of general solutions:

If $b^2 - 4ac > 0$ we guessed e^{rt} is a solution and noted that any linear combination of solutions is a solution to a homogeneous linear differential equation.

Section 3.3: If $b^2 - 4ac < 0$, :

Changed format of $y = c_1e^{r_1t} + c_2e^{r_2t}$ to linear combination of real-valued functions instead of complex valued functions by using Euler's formula:

$$e^{it} = cos(t) + isin(t)$$

Hence $e^{(d+in)t} = e^{dt}e^{int} = e^{dt}[cos(nt) + isin(nt)]$

Let $r_1 = d + in$, $r_2 = d - in$

$y = c_1e^{r_1t} + c_2e^{r_2t}$

$= c_1e^{dt}[cos(nt)+isin(nt)]+c_2e^{dt}[cos(-nt)+isin(-nt)]$

$= c_1e^{dt}cos(nt)+ic_1e^{dt}sin(nt)+c_2e^{dt}cos(nt)-ic_2e^{dt}sin(nt)$

$= (c_1 + c_2)e^{dt}cos(nt) + i(c_1 - c_2)e^{dt}sin(nt)$

$= k_1e^{dt}cos(nt) + k_2e^{dt}sin(nt)$

Section 3.4: If $b^2 - 4ac = 0$, then $r_1 = r_2$.
Hence one solution is $y = e^{r_1t}$ Need second solution.

If $y = e^{rt}$ is a solution, $y = ce^{rt}$ is a solution.

How about $y = v(t)e^{rt}$?

$y' = v'(t)e^{rt} + v(t)re^{rt}$

$y'' = v''(t)e^{rt} + v'(t)re^{rt} + v'(t)re^{rt} + v(t)r^2e^{rt}$

$= v''(t)e^{rt} + 2v'(t)re^{rt} + v(t)r^2e^{rt}$

$ay'' + by' + cy = 0$

$a(v''e^{rt}+2v're^{rt}+vr^2e^{rt})+b(v'e^{rt}+vre^{rt})+cve^{rt} = 0$

$a(v''(t)+2v'(t)r+v(t)r^2)+b(v'(t)+v(t)r)+cv(t) = 0$

$av''(t)+2av'(t)r+av(t)r^2+bv'(t)+bv(t)r+cv(t) = 0$

$av''(t) + (2ar + b)v'(t) + (ar^2 + br + c)v(t) = 0$

$av''(t) + (2a(\frac{-b}{2a}) + b)v'(t) + 0 = 0$

since $ar^2 + br + c = 0$ and $r = \frac{-b}{2a}$

$av''(t) + (-b + b)v'(t) = 0$. Thus $av''(t) = 0$.

Hence $v''(t) = 0$ and $v'(t) = k_1$ and $v(t) = k_1t + k_2$

Hence $v(t)e^{r_1t} = (k_1t + k_2)e^{r_1t}$ is a soln

Thus te^{r_1t} is a nice second solution.

Hence general solution is $y = c_1e^{r_1t} + c_2te^{r_1t}$

$d \pm ni$ $y = c_1(e^{dt}cos(nt)) + c_2(e^{dt}sin(nt))$

Solve: $y'' + y = 0$, $y(0) = -1$, $y'(0) = -3$ $\quad d = 0$

$r^2 + 1 = 0$ implies $r^2 = -1$. Thus $r = \pm i$. $\quad n = 1$

Since $r = 0 \pm 1i$, $\boxed{y = k_1cos(t) + k_2sin(t)}$. Then $y' = -k_1sin(t) + k_2cos(t)$

$y(0) = -1$: $-1 = k_1cos(0) + k_2sin(0)$ implies $-1 = k_1$

$y'(0) = -3$: $-3 = -k_1sin(0) + k_2cos(0)$ implies $-3 = k_2$

Thus IVP solution: $y = -cos(t) - 3sin(t)$

When does the following IVP have a unique solution:

IVP: $ay'' + by' + cy = 0$, $y(t_0) = y_0$, $y'(t_0) = y_1$.

Suppose $y = c_1\phi_1(t) + c_2\phi_2(t)$ is a solution to $ay'' + by' + cy = 0$. Then $y' = c_1\phi_1'(t) + c_2\phi_2'(t)$

$y(t_0) = y_0$: $y_0 = c_1\phi_1(t_0) + c_2\phi_2(t_0)$

$y'(t_0) = y_1$: $y_1 = c_1\phi_1'(t_0) + c_2\phi_2'(t_0)$

To find IVP solution, need to solve above system of two equations for the unknowns c_1 and c_2.

Note the IVP has a unique solution if and only if the above system of two equations has a unique solution for c_1 and c_2.

Note that in these equations c_1 and c_2 are the unknowns and $y_0, \phi_1(t_0), \phi_2(t_0)$, $y_1, \phi_1'(t_0), \phi_2'(t_0)$ are the constants. We can translate this linear system of equations into matrix form:

$$\begin{array}{l} c_1\phi_1(t_0) + c_2\phi_2(t_0) = y_0 \\ c_1\phi_1'(t_0) + c_2\phi_2'(t_0) = y_1 \end{array} \quad \text{implies} \quad \begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$$

Note this equation has a unique solution if and only if $det \begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) \end{bmatrix} = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} = \phi_1\phi_2' - \phi_1'\phi_2 \neq 0$

Definition: The Wronskian of two differential functions, ϕ_1 and ϕ_2 is

$$W(\phi_1, \phi_2) = \phi_1\phi_2' - \phi_1'\phi_2 = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}$$

Examples:

1.) Wronskian of $cos(t)$, $sin(t)$ $= \begin{vmatrix} cos(t) & sin(t) \\ -sin(t) & cos(t) \end{vmatrix} = cos^2(t) + sin^2(t) = 1 > 0.$ $\Rightarrow$ $cos\,t$ & $sin\,t$ are l. i

2.) Wronskian of $e^{dt}cos(nt)$, $e^{dt}sin(nt)$ $= \begin{vmatrix} e^{dt}cos(nt) & e^{dt}sin(nt) \\ de^{dt}cos(nt) - ne^{dt}sin(nt) & de^{dt}sin(nt) + ne^{dt}cos(nt) \end{vmatrix}$ deriv

$= e^{dt}cos(nt)[de^{dt}sin(nt) + ne^{dt}cos(nt)] - e^{dt}sin(nt)[de^{dt}cos(nt) - ne^{dt}sin(nt)]$

$= e^{2dt}(cos(nt)[dsin(nt) + ncos(nt)] - sin(nt)[dcos(nt) - nsin(nt)])$

$= e^{2dt}(dcos(nt)sin(nt) + ncos^2(nt)] - dsin(nt)cos(nt) + nsin^2(nt)])$

$= e^{2dt}(ncos^2(nt) + nsin^2(nt)) = ne^{2dt}(cos^2(nt) + sin^2(nt)) = ne^{2dt} > 0$ for all t.

$n, d \in \mathbb{R}$, $n > 0$

Need coef of 1 for y' for thm 2.4.1 AND also for solving when using integrating factor

Existence and Uniqueness

1st order LINEAR differential equation:

Thm 2.4.1: If $p : (a,b) \to R$ and $g : (a,b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\boxed{\phi : (a,b) \to R}$ that satisfies the initial value problem

$$\boxed{1 y' + p(t)y = g(t),}$$
$$y(t_0) = y_0$$

Looked for max interval containing t_0 st p, g cont

2nd order LINEAR differential equation:

Thm 3.2.1: If $p : (a,b) \to R$, $q : (a,b) \to R$, and $g : (a,b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\boxed{\phi : (a,b) \to R}$ that satisfies the initial value problem

$$\boxed{1 y'' + p(t)y' + q(t)y = g(t),}$$
$$y(t_0) = y_0, \quad y'(t_0) = y_0'$$

Look for max interval containing t_0 st p, q, g are cont

Definition: The Wronskian of two differential functions, f and g is

$$W(f,g) = fg' - f'g = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$$

Thm 3.2.4: Given (1) the hypothesis of thm 3.2.1
(2) ϕ_1 and ϕ_2 are 2 sol'ns to $y'' + p(t)y' + q(t)y = 0$ (*)
(3) $W(\phi_1, \phi_2)(t_0) \neq 0$, for some $t_0 \in (a,b)$,
then if f is a solution to (*), then $f = c_1\phi_1 + c_2\phi_2$ for some c_1 and c_2.

Need coeff of 1 for y'' for using thm 3.2.1 ONLY, don't need for solving

Thm 2.4.2: Suppose $z = f(t,y)$ and $z = \frac{\partial f}{\partial y}(t,y)$ are continuous on $(a,b) \times (c,d)$ and the point $(t_0, y_0) \in (a,b) \times (c,d)$, then there exists an interval $(t_0 - h, t_0 + h) \subset (a,b)$ such that there exists a unique function $y = \phi(t)$ defined on $(t_0 - h, t_0 + h)$ that satisfies the following initial value problem:

Non linear

$$y' = f(t,y), \quad y(t_0) = y_0.$$

Note the initial value problem

$$y' = y^{\frac{1}{3}}, \; y(0) = 0 \qquad y(3) = 0$$

has an infinite number of different solutions.

$$y^{-\frac{1}{3}} dy = dt$$
$$\tfrac{3}{2} y^{\frac{2}{3}} = t + C$$
$$\boxed{y = \pm(\tfrac{2}{3}t + C)^{\frac{3}{2}}}$$
$$y(0) = 0 \text{ implies } C = 0$$

Thus $y = \pm(\frac{2}{3}t)^{\frac{3}{2}}$ are solutions.

$\boxed{y = 0}$ is also a solution, etc.

Compare to Thm 2.4.2:
$f(t,y) = y^{\frac{1}{3}}$ is continuous near $(0,0)$
But $\frac{\partial f}{\partial y}(t,y) = \frac{1}{3} y^{\frac{-2}{3}}$ is not continuous near $(0, 0)$ since it isn't defined at $(0, 0)$.