Ch 3

Second order differential equation:

Linear equation with constant coefficients:
If the second order differential equation is

$$ay'' + by' + cy = 0,$$

then $y = e^{rt}$ is a solution

Need to have two independent solutions.

Solve the following IVPs:

1.) $y'' - 6y' + 9y = 0$ $\qquad y(0) = 1,\ y'(0) = 2$

2.) $4y'' - y' + 2y = 0$ $\qquad y(0) = 3,\ y'(0) = 4$

3.) $4y'' + 4y' + y = 0$ $\qquad y(0) = 6,\ y'(0) = 7$

4.) $2y'' - 2y = 0$ $\qquad y(0) = 5,\ y'(0) = 9$

2 different real soln

$\boxed{ay'' + by' + cy = 0}$, $\boxed{y = e^{rt}}$, then
$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$ implies $ar^2 + br + c = 0$, characteristic eq'n.
Suppose $r = r_1, r_2$ are solutions to $ar^2 + br + c = 0$
$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $r_1 \neq r_2$, then $b^2 - 4ac \neq 0$. Hence a general solution is $y = c_1e^{r_1t} + c_2e^{r_2t}$ 2 real soln to

real #'s

3.1 If $b^2 - 4ac > 0$, general solution $\boxed{\text{is } y = c_1e^{r_1t} + c_2e^{r_2t}}$.

If $b^2 - 4ac < 0$, change format to linear combination of real-valued functions instead of complex valued functions by using Euler's formula.

3.3 2 different complex soln

$y = c_1 e^{(d+in)t} + c_2 e^{(d-in)t}$ ugly unaccep

general solution is $\boxed{y = c_1e^{dt}cos(nt) + c_2e^{dt}sin(nt)}$
where $r = d \pm in$

beautiful acceptable sol

If $b^2 - 4ac = 0$, $r_1 = r_2$, so need 2nd (independent) solution: te^{r_1t}

3.4: 1 real soln: e^{rt}

Hence general solution is $y = c_1e^{r_1t} + c_2te^{r_1t}$.

Initial value problem: use $y(t_0) = y_0$, $y'(t_0) = y'_0$ to solve for c_1, c_2 to find unique solution.

Ch 3

Derivation of general solutions:

3.1 If $b^2 - 4ac > 0$ we guessed e^{rt} is a solution and noted that any linear combination of solutions is a solution to a homogeneous linear differential equation.

Section 3.3: If $b^2 - 4ac < 0$, :

Changed format of $y = c_1e^{r_1t} + c_2e^{r_2t}$ to linear combination of real-valued functions instead of complex valued functions by using Euler's formula:

ugly when r_1, r_2 are complex

$$\boxed{e^{it} = cos(t) + isin(t)}$$

Hence $e^{(d+in)t} = e^{dt}e^{int} = e^{dt}[cos(nt) + isin(nt)]$

Let $r_1 = d + in$, $r_2 = d - in$

$y = c_1e^{r_1t} + c_2e^{r_2t}$ ← ugly

$= c_1e^{dt}[cos(nt)+isin(nt)]+c_2e^{dt}[cos(-nt)+isin(-nt)]$

$= c_1e^{dt}cos(nt)+ic_1e^{dt}sin(nt)+c_2e^{dt}cos(nt)-ic_2e^{dt}sin(nt)$

$=(c_1 + c_2)e^{dt}cos(nt) + i(c_1 - c_2)e^{dt}sin(nt)$

$y = k_1e^{dt}cos(nt) + k_2e^{dt}sin(nt)$ ← acceptable

Section 3.4: If $b^2 - 4ac = 0$, then $r_1 = r_2$.
Hence one solution is $y = e^{r_1t}$ Need second solution.

If $y = e^{rt}$ is a solution, $y = ce^{rt}$ is a solution.

How about $\boxed{y = v(t)e^{rt}?}$

$y' = v'(t)e^{rt} + v(t)re^{rt}$

$y'' = v''(t)e^{rt} + v'(t)re^{rt} + v'(t)re^{rt} + v(t)r^2e^{rt}$

$= v''(t)e^{rt} + 2v'(t)re^{rt} + v(t)r^2e^{rt}$

$ay'' + by' + cy = 0$

$a(v''e^{rt}+2v're^{rt}+vr^2e^{rt})+b(v'e^{rt}+vre^{rt})+cve^{rt} = 0$

$a(v''(t)+2v'(t)r+v(t)r^2)+b(v'(t)+v(t)r)+cv(t) = 0$

$av''(t)+2av'(t)r+av(t)r^2+bv'(t)+bv(t)r+cv(t) = 0$

$av''(t) + (2ar + b)v'(t) + (ar^2 + br + c)v(t) = 0$

$av''(t) + (2a(\frac{-b}{2a}) + b)v'(t) + 0 = 0$

since $ar^2 + br + c = 0$ and $r = \frac{-b}{2a}$ ← e^{rt} soln!

$av''(t) + (-b + b)v'(t) = 0.$ Thus $av''(t) = 0.$

Hence $v''(t) = 0$ and $v'(t) = k_1$ and $v(t) = k_1t + k_2$

Hence $v(t)e^{r_1t} = (k_1t + k_2)e^{r_1t}$ is a soln

Thus te^{r_1t} is a nice second solution.

Hence general solution is $\boxed{y = c_1e^{r_1t} + c_2te^{r_1t}}$

one repeated root

Solving first order differential equation:

Method 1 (sect. 2.2): Separate variables.

Method 2 (sect. 2.1): If linear $[y'(t)+p(t)y(t) = g(t)]$, multiply equation by an integrating factor $u(t) = e^{\int p(t)dt}$

Ch 2. basics

$$
\begin{gathered}
1\ y' + py = g \\
y'u + upy = ug \\
(uy)' = ug \\
\int (uy)' = \int ug \\
uy = \int ug \\
\text{etc...}
\end{gathered}
$$

Method 3 (sect. 2.4): Solve Bernoulli's equation,

$$y' + p(t)y = g(t)y^n,$$

when $n > 1$ by changing it to a linear equation by substituting $v = y^{1-n}$

If $v = \frac{dx}{dt}$, can use the following to simplify (especially if there are 3 variables).

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

integration techniques: u-substitution, integration by parts, partial fractions.

---

direction field = slope field = graph of $\frac{dv}{dt}$ in t, v-plane.

*** can use slope field to determine behavior of v including as $t \to \infty$.

Equilibrium Solution = constant solution

stable, unstable, semi-stable.

Solving second order differential equation:

p. 135: $y'' = f(t, y')$, $y'' = f(y, y')$,

Transform to first order: Let $v = y'$.

If needed, note $v' = \frac{dv}{dt} = \frac{dv}{dt}\frac{dy}{dy} = \frac{dv}{dy}\frac{dy}{dt} = \frac{dv}{dy}v$.

Note this trick sometimes helpful for first order equations.

Ch 3: linear $ay'' + by' + cy = 0$,

Need to have two independent solutions.

If ϕ_1, ϕ_2 are solutions to a LINEAR HOMOGENEOUS differential equation, $c_1\phi_1 + c_2\phi_2$ is also a solution

$3.1 \Rightarrow y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

IVP

Thm 2.4.2 $y' = f(t, y)$

If f cont? Is $\frac{\partial f}{\partial y}$ cont?

Existence and Uniqueness

1st order LINEAR differential equation:

Thm 2.4.1: If $p : (a, b) \to R$ and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a, b) \to R$ that satisfies the initial value problem

$$\begin{aligned} y' + p(t)y &= g(t), \\ y(t_0) &= y_0 \end{aligned}$$

2nd order LINEAR differential equation:

Thm 3.2.1: If $p : (a, b) \to R$, $q : (a, b) \to R$, and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a, b) \to R$ that satisfies the initial value problem

$$\begin{aligned} y'' + p(t)y' + q(t)y &= g(t), \\ y(t_0) &= y_0, \\ y'(t_0) &= y'_0 \end{aligned}$$

Thm 3.2.2: If ϕ_1 and ϕ_2 are two solutions to a homogeneous linear differential equation, the $c_1\phi_1 + c_2\phi_2$ is also a solution to this linear differential equation.

Definition: The Wronskian of two differential functions, f and g is

$$W(f, g) = fg' - f'g = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$$

Thm 3.2.3: Suppose that ϕ_1 and ϕ_2 are two solutions to $y'' + p(t)y' + q(t)y = 0$. If $W(\phi_1, \phi_2)(t_0) = \phi_1(t_0)\phi_2'(t_0) - \phi_1'(t_0)\phi_2(t_0) \neq 0$, then there is a unique choice of constants c_1 and c_2 such that $c_1\phi_1 + c_2\phi_2$ satisfies this homogeneous linear differential equation and initial conditions, $y(t_0) = y_0$, $y'(t_0) = y'_0$.

Thm 3.2.4: Given the hypothesis of thm 3.2.1 Suppose that ϕ_1 and ϕ_2 are two solutions to $y'' + p(t)y' + q(t)y = 0$. If $W(\phi_1, \phi_2)(t_0) \neq 0$, for some $t_0 \in (a, b)$, then any solution to this homogeneous linear differential equation can be written as $y = c_1\phi_1 + c_2\phi_2$ for some c_1 and c_2.

Defn If ϕ_1 and ϕ_2 satisfy the conditions in thm 3.2.4, then ϕ_1 and ϕ_2 form a fundamental set of solutions to $y'' + p(t)y' + q(t)y = 0$.

Thm 3.2.5: Given any second order homogeneous linear differential equation, there exist a pair of functions which form a fundamental set of solutions.

Existence and Uniqueness

1st order LINEAR differential equation:

Thm 2.4.1: If $p : (a, b) \to R$ and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a, b) \to R$ that satisfies the initial value problem

$$y' + p(t)y = g(t),$$
$$y(t_0) = y_0$$

2nd order LINEAR differential equation:

Thm 3.2.1: If $p : (a, b) \to R$, $q : (a, b) \to R$, and $g : (a, b) \to R$ are continuous and $a < t_0 < b$, then there exists a unique function $y = \phi(t)$, $\phi : (a, b) \to R$ that satisfies the initial value problem

$$y'' + p(t)y' + q(t)y = g(t),$$
$$y(t_0) = y_0, \quad y'(t_0) = y'_0$$

later

Definition: The Wronskian of two differential functions, f and g is

$$W(f, g) = fg' - f'g = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}$$

Thm 3.2.4: Given (1) the hypothesis of thm 3.2.1
(2) ϕ_1 and ϕ_2 are 2 sol'ns to $y'' + p(t)y' + q(t)y = 0$ (*)
(3) $W(\phi_1, \phi_2)(t_0) \neq 0$, for some $t_0 \in (a, b)$,
then if f is a solution to (*), then $f = c_1\phi_1 + c_2\phi_2$ for some c_1 and c_2.

Thm 2.4.2: Suppose $z = f(t, y)$ and $z = \frac{\partial f}{\partial y}(t, y)$ are continuous on $(a, b) \times (c, d)$ and the point $(t_0, y_0) \in (a, b) \times (c, d)$, then there exists an interval $(t_0 - h, t_0 + h) \subset (a, b)$ such that there exists a unique function $y = \phi(t)$ defined on $(t_0 - h, t_0 + h)$ that satisfies the following initial value problem:

$$y' = f(t, y), \quad y(t_0) = y_0.$$

Note the initial value problem

$$y' = y^{\frac{1}{3}}, \; y(0) = 0$$

has an infinite number of different solutions.

$$y^{-\frac{1}{3}}dy = dt$$
$$\frac{3}{2}y^{\frac{2}{3}} = t + C$$
$$y = \pm(\tfrac{2}{3}t + C)^{\frac{3}{2}}$$
$$y(0) = 0 \text{ implies } C = 0$$

Thus $y = \pm(\frac{2}{3}t)^{\frac{3}{2}}$ are solutions.

$y = 0$ is also a solution, etc.

Compare to Thm 2.4.2:
$f(t, y) = y^{\frac{1}{3}}$ is continuous near (0, 0)
But $\frac{\partial f}{\partial y}(t, y) = \frac{1}{3}y^{\frac{-2}{3}}$ is not continuous near (0, 0) since it isn't defined at (0, 0).

2.3

Suppose salty water enters and leaves a tank at a rate of 2 liters/minute.

Suppose also that the salt concentration of the water entering the tank varies with respect to time according to $Q(t) \cdot tsin(t^2)$ g/liters where $Q(t)$ = amount of salt in tank in grams. (Note: this is not realistic).

If the tank contains 4 liters of water and initially contains 5g of salt, find a formula for the amount of salt in the tank after t minutes.

check units to see if rate equation is correct

Let $Q(t)$ = amount of salt in tank in grams.

$\frac{dQ}{dt} = g/min$

Note $Q(0) = 5$ g

rate in $= (2 \text{ liters/min})(Q(t) \cdot tsin(t^2) \text{ g/liters})$

$= 2Qtsin(t^2)$ g/min

rate out $= (2 \text{ liters/min})(\frac{Q(t)g}{4\text{liters}}) = \frac{Q}{2}$ g/min

$\boxed{\frac{dQ}{dt} = \text{rate in - rate out}} = 2Qtsin(t^2) - \frac{Q}{2}$

$\frac{dQ}{dt} = Q(2tsin(t^2) - \frac{1}{2})$

This is a first order linear ODE. It is also a separable ODE. Thus can use either 2.1 or 2.2 methods.

Note always, but most of our examples where of format $\frac{dQ}{dt}$ = rate in − rate out
Eg: (Salt in, salt out) and (\$ in, \$ out)

Using the easier 2.2:

$\int \frac{dQ}{Q} = \int (2tsin(t^2) - \frac{1}{2})dt = \int 2tsin(t^2)dt - \int \frac{1}{2}dt$

Let $u = t^2$, $du = 2tdt$

$ln|Q| = \int sin(u)du - \frac{t}{2} = -cos(u) - \frac{t}{2} + C$
$= -cos(t^2) - \frac{t}{2} + C$

$|Q| = e^{-cos(t^2) - \frac{t}{2} + C} = e^C e^{-cos(t^2) - \frac{t}{2}}$

$Q = Ce^{-cos(t^2) - \frac{t}{2}}$

$Q(0) = 5: \quad 5 = Ce^{-1-0} = Ce^{-1}$. Thus $C = 5e$

Thus $Q(t) = 5e \cdot e^{-cos(t^2) - \frac{t}{2}}$

Thus $Q(t) = 5e^{-cos(t^2) - \frac{t}{2} + 1}$

Long-term behaviour:

$Q(t) = 5(e^{-cos(t^2)})(e^{\frac{-t}{2}})e$

As $t \to \infty$, $e^{\frac{-t}{2}} \to 0$, while $5(e^{-cos(t^2)})e$ are finite.

Thus as $t \to \infty$, $Q(t) \to 0$.

IVP $y' = \frac{1}{(1-t)(2-y)}$ $y(t_0) = y_0$

Section 2.4 example: $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$

$F(y,t) = \frac{1}{(1-t)(2-y)}$ is continuous for all $t \neq 1$, $y \neq 2$

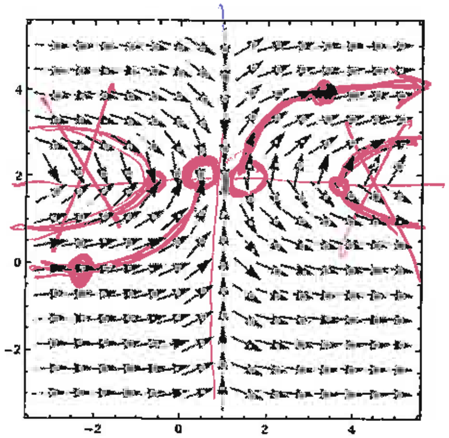
$\frac{\partial F}{\partial y} = \frac{\partial\left(\frac{1}{(1-t)(2-y)}\right)}{\partial y} = \frac{1}{(1-t)} \frac{\partial (2-y)^{-1}}{\partial y} = \frac{1}{(1-t)(2-y)^2}$

$\frac{\partial F}{\partial y}$ is continuous for all $t \neq 1$, $y \neq 2$

Thus the IVP $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$, $y(t_0) = y_0$ has a unique solution if $t_0 \neq 1$, $y_0 \neq 2$.

Note that if $y_0 = 2$, $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$, $y(t_0) = 2$ has two solutions if $t_0 \neq 1$ if allow one sided derivatives

Note that if $t_0 = 1$, $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$, $y(1) = y_0$ has no solutions.



$(1, 1/((1-x)(2-y)))/sqrt1 + 1/((1-x)(2-y))^2)$

Solve via separation of variables:

$t_0 \neq 1$
$y_0 \neq 2$
$\Downarrow$
unique sol

$\int (2-y)dy = \int \frac{dt}{1-t}$

$2y - \frac{y^2}{2} = -ln|1-t| + C$

$y^2 - 4y - 2ln|1-t| + C = 0$

$y = \frac{4 \pm \sqrt{16 + 4(2ln|1-t| + C)}}{2} = 2 \pm \sqrt{4 + 2ln|1-t| + C}$

$$y = 2 \pm \sqrt{2ln|1-t| + C}$$

Find domain: $2ln|1-t| + C \geq 0$

$2ln|1-t| \geq -C$

$ln|1-t| \geq -\frac{C}{2}$ Note: we want to find domain for this C and thus this C can't swallow constants).

$|1-t| \geq e^{-\frac{C}{2}}$ since e^x is an increasing function.

$1 - t \leq -e^{-\frac{C}{2}}$ or $1 - t \geq e^{-\frac{C}{2}}$

$-t \leq -e^{-\frac{C}{2}} - 1$ or $-t \geq e^{-\frac{C}{2}} - 1$

Domain: $\begin{cases} t \geq e^{-\frac{C}{2}} + 1 & \text{if } t_0 > 0 \\ t \leq -e^{-\frac{C}{2}} + 1 & \text{if } t_0 < 0. \end{cases}$

Note: Domain is much easier to determine when the ODE is linear.

not allowing 1-sided derivatives
Thus no closed intervals for domains

Find C given $y(t_0) = y_0$**:** $y_0 = 2 \pm \sqrt{2ln|1-t_0| + C}$

$\pm(y_0 - 2) = \sqrt{2ln|1-t_0| + C}$

$(y_0 - 2)^2 - 2ln|1-t_0| = C$

$y = 2 \pm \sqrt{2ln|1-t| + C}$

$y = 2 \pm \sqrt{2ln|1-t| + (y_0-2)^2 - 2ln|1-t_0|}$

$y = 2 \pm \sqrt{(y_0-2)^2 + ln\frac{(1-t)^2}{(1-t_0)^2}}$

Domain: $\begin{cases} t \geq e^{-\frac{C}{2}} + 1 & \text{if } t_0 > 0 \\ t \leq -e^{-\frac{C}{2}} + 1 & \text{if } t_0 < 0. \end{cases}$

$e^{-\frac{C}{2}} = e^{-\frac{(y_0-2)^2 - 2ln|1-t_0|}{2}} = |1-t_0|e^{-\frac{(y_0-2)^2}{2}}$

Domain: $\begin{cases} t \geq 1 + |1-t_0|e^{-\frac{(y_0-2)^2}{2}} & \text{if } t_0 > 0 \\ t \leq 1 - |1-t_0|e^{-\frac{(y_0-2)^2}{2}} & \text{if } t_0 < 0. \end{cases}$

Section 2.5: Autonomous eqn

$y' = f(y)$

Exponential Growth/Decay
Example: population growth/radioactive decay)

$y' = ry$, $y(0) = y_0$ implies $y = y_0e^{rt}$

$y' = f(y)$

$r > 0$ $r < 0$

Logistic growth: $y' = h(y)y$

Example: $y' = r(1 - \frac{y}{K})y$

y vs $f(y)$ slope field:

Equilibrium solutions:

Asymptotically stable:

Asymptotically unstable:

Asymptotically semi-stable:

As $t \to \infty$, if $y > 0$, $y \to$

Solution: $y = \frac{y_0K}{y_0 + (K - y_0)e^{-rt}}$

Section 2.7 Euler method: Using tangent lines to approximate a function.

$y_{i+1} = y_i + \Delta y = y_i + \frac{\Delta y}{\Delta t}\Delta t \cong y_i + \frac{dy}{dt}\Delta t$

Alternatively use equation of tangent line:

$$\text{slope} = \frac{y_{i+1}-y_i}{t_{i+1}-t_i} = f'(y_i,t_i).$$

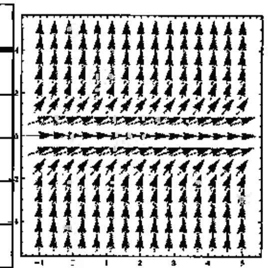
$y_{i+1} = f'(y_i,t_i)(t_{i+1}-t_i)+y_i = T(t_{i+1})$ where $y = T(t)$ is the equation of the tangent line at (y_i,t_i).

Example: $\frac{dy}{dt} = y^2$, $y(2) = 1$ implies $y = \frac{1}{3-t}$.

t	$y = 1/(3-t)$	approximation
2.000000	1.000000	1.000000
3.000000	999.000000	2.000000
4.000000	-1.000000	6.000000
5.000000	-0.500000	42.000000
6.000000	-0.333333	1806.000000

t	$y = 1/(3-t)$	approximation
2.000000	1.000000	1.000000
2.100000	1.111111	1.100000
2.200000	1.250000	1.221000
2.300000	1.428571	1.370084
2.400000	1.666667	1.557797
2.500000	2.000000	1.800470
2.600000	2.500000	2.124640
2.700000	3.333333	2.576049
2.800000	5.000000	3.239652
2.900000	10.000004	4.289186

t	$y = 1/(3-t)$	approximation
2.00	1.000000	1.000000
2.01	1.010101	1.010000
2.02	1.020408	1.020201
2.03	1.030928	1.030609
2.04	1.041667	1.041231
2.05	1.052632	1.052072
2.06	1.063830	1.063141
2.07	1.075269	1.074443
2.08	1.086957	1.085988
2.09	1.098901	1.097782
2.10	1.111111	1.109833
2.11	1.123595	1.122150
2.12	1.136364	1.134742
2.13	1.149425	1.147619
2.87	7.692308	6.721314
2.88	8.333333	7.173075
2.89	9.090908	7.687605
2.90	9.999998	8.278598
2.91	11.111107	8.963949
2.92	12.499993	9.767473
2.93	14.285716	10.721509
2.94	16.666666	11.871017
2.95	19.999996	13.280227
2.96	24.999987	15.043871
2.97	33.333298	17.307051
2.98	49.999897	20.302391
2.99	99.999496	24.424261



$y' = y^2$

PRO (http://www.wolframalpha.com/pro) ABOUT (//www.wolframalpha.co

/pro?src=pro-banner-5)

slope field

Examples (//www.wolframalpha.com/examples/?src=input) Random

Assuming "slope field" refers to a computation | Use as referring to a mathematical definition instead

vector field: {1,x + 2y}/sqrt(1 + (x

variable 1: x

lower limit 1: -.1

upper limit 1: 0.4

variable 2: y

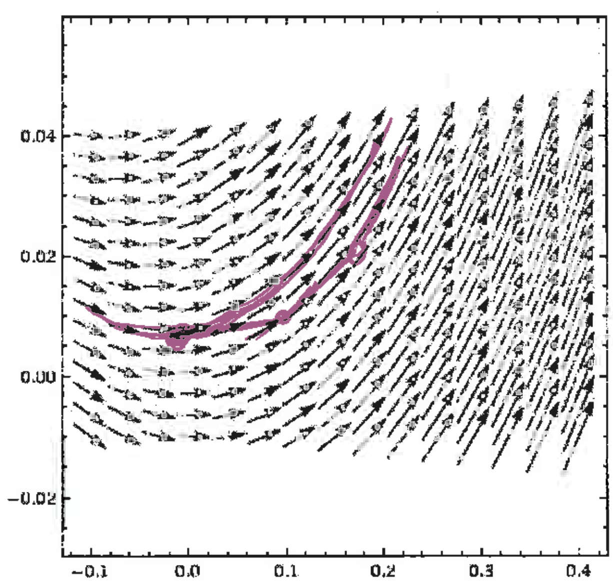
lower limit 2: -.01

upper limit 2: 0.04

Input:

$$\text{VectorPlot}\left[\frac{\{1, x+2y\}}{\sqrt{1+(x+2y)^2}}, \{x, -0.1, 0.4\}, \{y, -0.01, 0.04\}\right]$$

Result:



2.7
piece together a bunch of tangent lines to approx soln

2.8

Given: $y' = f(t, y)$, $y(0) = 0$ Eqn (*)

f, $\partial f/\partial y$ continuous $\forall (t, y) \in (-a, a) \times (-b, b)$. Then

$\Rightarrow$ $y = \phi(t)$ is a solution to (*) iff

$\phi'(t) = f(t, \phi(t))$, $\phi(0) = 0$ iff

$\int_0^t \phi'(s)ds = \int_0^t f(s, \phi(s)ds$, $\phi(0) = 0$ iff

$\phi(t) = \phi(t) - \phi(0) = \int_0^t f(s, \phi(s)ds$

Thus $y = \phi(t)$ is a solution to (*) iff $\phi(t) = \int_0^t f(s, \phi(s)ds$

Construct ϕ using method of successive approximation – also called Picard's iteration method.

Let $\phi_0(t) = 0$ (or the function of your choice)

Let $\phi_1(t) = \int_0^t f(s, \phi_0(s)ds$

Let $\phi_2(t) = \int_0^t f(s, \phi_1(s)ds$

Another way to approximate soln to ODE

$\vdots$

Let $\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s)ds$

Let $\phi(t) = \lim_{n\to\infty} \phi_n(t)$

Some questions:

1.) Does $\phi_n(t)$ exist for all n?

2.) Does sequence ϕ_n converge?

3.) Is $\phi(t) = \lim_{n\to\infty} \phi_n(t)$ a solution to (*).

4.) Is the solution unique.

Example: $y' = t + 2y$. That is $f(t, y) = t + 2y$

Let $\phi_0(t) = 0$

Let $\phi_1(t) = \int_0^t f(s, 0)ds = \int_0^t (s + 2(0))ds$

$$= \int_0^t s\,ds = \frac{s^2}{2}\Big|_0^t = \frac{t^2}{2}$$

Let $\phi_2(t) = \int_0^t f(s, \phi_1(s))ds = \int_0^t f(s, \frac{s^2}{2})ds$

$$= \int_0^t (s + 2(\frac{s^2}{2}))ds = \frac{t^2}{2} + \frac{t^3}{3}$$

Let $\phi_3(t) = \int_0^t f(s, \phi_2(s))ds = \int_0^t f(s, \frac{s^2}{2} + \frac{s^3}{3})ds$

$$= \int_0^t (s + 2\frac{s^2}{2} + \frac{s^3}{3}))ds = \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6}$$

$\vdots$

See class notes.