2.5: $y' = f(y)$ ~ autonomous ODE

2.) Circle the differential equation whose direction field is given below:

A) $y' = t^2$ B) $y' = \frac{1}{2}$

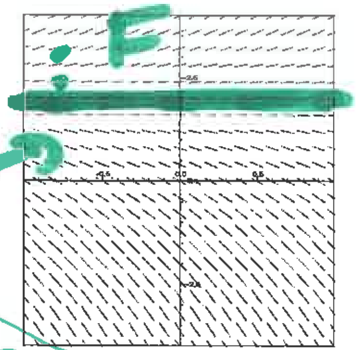
C) $y' = 1$ D) $y' = -1$ Equil solns

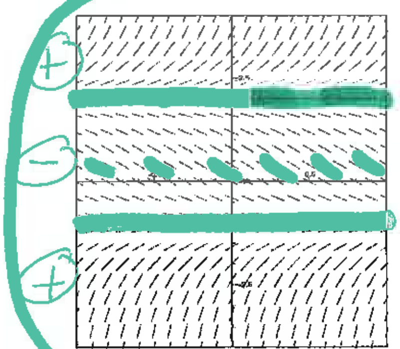
E) $y' = y + 1$ F) $y' = y - 2$ $\Rightarrow y = 2$

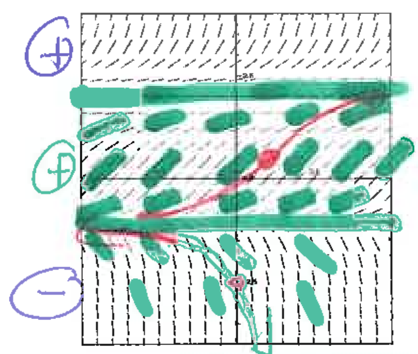
G) $y' = (y+1)(y-2)$ H) $y' = (y+1)^2(y-2)^2$

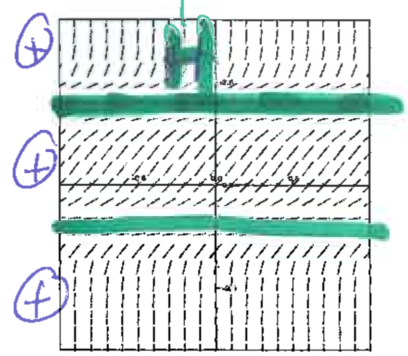
I) $y' = (y+1)(y-2)^2$ J) $y' = (y+1)^2(y-2)$

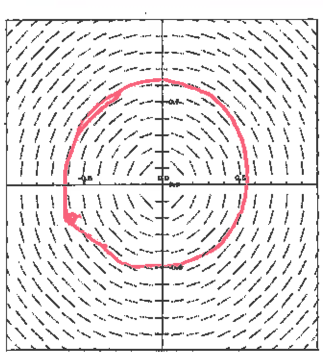
K) $y = -\frac{t}{y}$ $y = -1, y = 2$

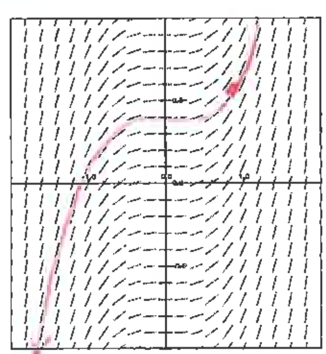


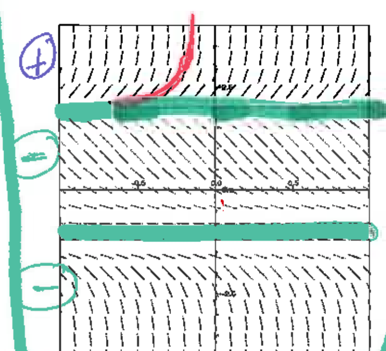










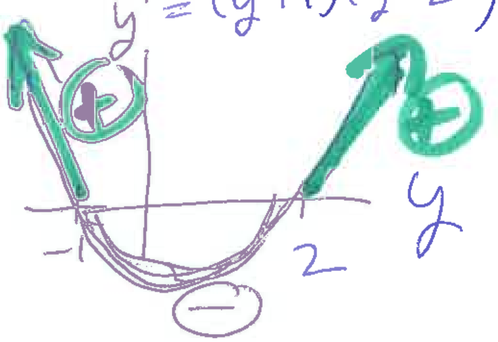


$y' = (y+1)(y-2)$

$\oplus$ $\ominus$ $\oplus$

-1 2

$y' = (y+1)(y-2) = f(y)$



$\int_0^t \phi'(s)ds = \phi(s)\big|_0^t$

Given: $y' = f(t,y),\ y(0) = 0$ Eqn (*)

$f,\ \partial f/\partial y$ continuous $\forall (t,y) \in (-a,a) \times (-b,b)$. Then

$y = \phi(t)$ is a solution to (*) iff

$\boxed{\phi'(t) = f(t, \phi(t)), \quad \phi(0) = 0}$ iff

$\int_0^t \phi'(s)ds = \int_0^t f(s, \phi(s)ds, \quad \phi(0) = 0$ iff

$\phi(t) = \phi(t) - \phi(0) = \int_0^t f(s, \phi(s)ds$

Thus $y = \phi(t)$ is a solution to (*) iff $\phi(t) = \int_0^t f(s, \phi(s)ds$

Construct ϕ using method of successive approximation – also called Picard's iteration method.

Create a sequence of functions

Let $\phi_0(t) = 0$ (or the function of your choice)

Let $\phi_1(t) = \int_0^t f(s, \phi_0(s)ds$

Let $\phi_2(t) = \int_0^t f(s, \phi_1(s)ds$

⋮

Let $\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s)ds$

Let $\phi(t) = \lim_{n\to\infty} \phi_n(t)$

Some questions:

1.) Does $\phi_n(t)$ exist for all n?

2.) Does sequence ϕ_n converge?

3.) Is $\phi(t) = \lim_{n\to\infty} \phi_n(t)$ a solution to (*).

4.) Is the solution unique.

Example: $y' = t + 2y$. That is $f(t,y) = t + 2y$

cont $\forall\ t, y$

$\frac{\partial f}{\partial y} = 2$ cont $\forall t, y$

Let $\phi_0(t) = 0$

Let $\phi_1(t) = \int_0^t f(s, 0)ds = \int_0^t (s + 2(0))ds$

$= \int_0^t s\,ds = \frac{s^2}{2}\big|_0^t = \frac{t^2}{2}$

Let $\phi_2(t) = \int_0^t f(s, \phi_1(s))ds = \int_0^t f(s, \frac{s^2}{2})ds$

$= \int_0^t (s + 2(\frac{s^2}{2}))ds = \frac{t^2}{2} + \frac{t^3}{3}$

Let $\phi_3(t) = \int_0^t f(s, \phi_2(s))ds = \int_0^t f(s, \frac{s^2}{2} + \frac{s^3}{3})ds$

$= \int_0^t (s + 2\frac{s^2}{2} + \frac{s^3}{3}))ds = \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6}$

⋮

See class notes.

$$\phi_4(t) = \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6} + \frac{t^5}{15} = \sum_{k=1}^{4} \frac{t^{k+1}}{(k+1)!}$$

$$\boxed{\phi_n(t) = \sum_{k=1}^{n} \frac{2^{k-1}\, t^{k+1}}{(k+1)!}}$$

$$\frac{t^2}{2} + \frac{2t^3}{3\cdot 2} + \frac{4t^4}{4\cdot 6} + \frac{2^3 t^5}{2^3 15} = \frac{t^2}{2!} + \frac{2t^3}{3!} + \frac{2^2 t^4}{4!} + \frac{2^3 t^5}{5!}$$

$4\cdot 3\cdot 2$ $\quad$ $5\cdot 4\cdot 3\cdot 2\cdot 1$

$$\frac{2^{k-1}\, t^{k+1}}{(k+1)!}$$