2.4 Solve Bernoulli's equation,

$$y' + p(t)y = g(t)y^n,$$

when $n \neq 0, 1$ by changing it

$$y^{-n}y' + p(t)y^{1-n} = g(t)$$

when $n \neq 0, 1$ by changing it to a linear equation by substituting $v = y^{1-n}$

Solve $ty' + 2t^{-2}y = 2t^{-2}y^5$

$ty^{-5}y' + 2t^{-2}y^{-4} = 2t^{-2}$

Let $v = y^{-4}$. Thus $v' = \underline{-4y^{-5}y'}$

$-4ty^{-5}y' - 8t^{-2}y^{-4} = -8t^{-2}$

$tv' - 8t^{-2}v = -8t^{-2}$ ← linear ODE

Make coefficient of $v' = 1$

$v' - 8t^{-3}v = -8t^{-3}$

An antiderivative of $-8t^{-3}$ is $4t^{-2}$

Multiply equation by $e^{4t^{-2}}$

$e^{4t^{-2}}v' - 8t^{-3}e^{4t^{-2}}v = -8t^{-3}e^{4t^{-2}}$

$(e^{4t^{-2}}v)'$

$(e^{4t^{-2}}v)' = -8t^{-3}e^{4t^{-2}}$ by PRODUCT rule.

$\int (e^{4t^{-2}}v)'dt = -8\int t^{-3}e^{4t^{-2}}dt$

$e^{4t^{-2}}v = -8\int t^{-3}e^{4t^{-2}}dt.$

Let $u = 4t^{-2}$. Then $du = -8t^{-3}dt$

$e^{4t^{-2}}v = \int e^u du = e^u + C$

$e^{4t^{-2}}v = e^{4t^{-2}} + C$

$v = 1 + Ce^{-4t^{-2}}$

final answer in terms of y & t

$y^{-4} = 1 + Ce^{-4t^{-2}}$ implies $\boxed{y = \pm(1 + Ce^{-4t^{-2}})^{-\frac{1}{4}}}$

$y' + \frac{2}{t-3}y = 1$

An anti-derivative of $\frac{2}{t-3} = 2ln(t-3)$

$e^{2ln(t-3)} = e^{ln[(t-3)^2]} = (t-3)^2$

$y' + \frac{2}{t-3}y = 1$

$(t-3)^2y' + 2(t-3)y = (t-3)^2$

$\int [(t-3)^2y]' = \int (t-3)^2$

$(t-3)^2y = \frac{(t-3)^3}{3} + C$ implies $y = \frac{(t-3)}{3} + C(t-3)^{-2}$

Does a unique soln exist to IVP ~~involving~~ involving

Section 2.4 example: $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$

$F(y,t) = \frac{1}{(1-t)(2-y)}$ is continuous for all $\boxed{t \neq 1,\ y \neq 2}$

$\frac{\partial F}{\partial y} = \frac{\partial\left(\frac{1}{(1-t)(2-y)}\right)}{\partial y} = \frac{1}{(1-t)}\frac{\partial(2-y)^{-1}}{\partial y} = \frac{1}{(1-t)(2-y)^2}$

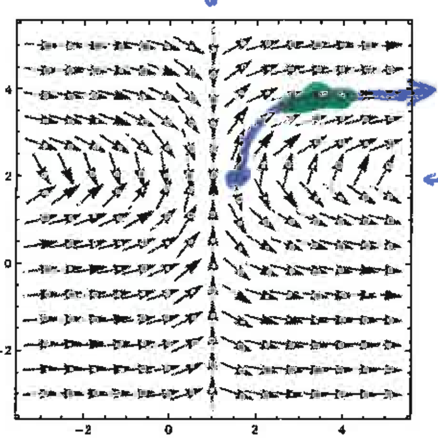
$\frac{\partial F}{\partial y}$ is continuous for all $\boxed{t \neq 1,\ y \neq 2}$

Thus the IVP $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$, $y(t_0) = y_0$ has a unique solution if $t_0 \neq 1$, $y_0 \neq 2$.

Note that if $y_0 = 2$, $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$, $y(t_0) = 2$ has two solutions if $t_0 \neq 2$

Note that if $t_0 = 1$, $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$, $y(1) = y_0$ has no solutions.

$t = 1$



$y = 2$

$(1, 1/((1-x)(2-y)))/sqrt1 + 1/((1-x)(2-y))^2)$

$\frac{dy}{dt} = y' = \frac{1}{(1-t)(2-y)}$

Solve via separation of variables:

$\int (2-y)dy = \int \frac{dt}{1-t}$

$2y - \frac{y^2}{2} = -ln|1-t| + C$

$y^2 - 4y - 2ln|1-t| + C = 0$

$y = \frac{4 \pm \sqrt{16+4(2ln|1-t|+C)}}{2} = 2 \pm \sqrt{4 + 2ln|1-t| + C}$

$$\boxed{y = 2 \pm \sqrt{2ln|1-t| + C}}$$

Find domain: $2ln|1-t| + C \geq 0$

$2ln|1-t| \geq -C$

$ln|1-t| \geq -\frac{C}{2}$ Note: we want to find domain for this C and thus this C can't swallow constants).

$|1-t| \geq e^{-\frac{C}{2}}$ since e^x is an increasing function.

$1 - t \leq -e^{-\frac{C}{2}}$ or $1 - t \geq e^{-\frac{C}{2}}$

$-t \leq -e^{-\frac{C}{2}} - 1$ or $-t \geq e^{-\frac{C}{2}} - 1$

Domain: $\begin{cases} t \geq e^{-\frac{C}{2}} + 1 & \text{if } t_0 > 0 \\ t \leq -e^{-\frac{C}{2}} + 1 & \text{if } t_0 < 0. \end{cases}$

C means the same thing in both expressions

Note: Domain is much easier to determine when the ODE is linear.

Find C given $y(t_0) = y_0$: $y_0 = 2 \pm \sqrt{2ln|1-t_0| + C}$

$\pm(y_0 - 2) = \sqrt{2ln|1-t_0| + C}$

$\boxed{(y_0-2)^2 - 2ln|1-t_0| = C}$

$y = 2 \pm \sqrt{2ln|1-t| + C}$

$y = 2 \pm \sqrt{2ln|1-t| + (y_0-2)^2 - 2ln|1-t_0|}$

$\boxed{y = 2 \pm \sqrt{(y_0-2)^2 + ln\frac{(1-t)^2}{(1-t_0)^2}}}$

Domain: $\begin{cases} t \geq e^{-\frac{C}{2}} + 1 & \text{if } t_0 > 0 \\ t \leq -e^{-\frac{C}{2}} + 1 & \text{if } t_0 < 0. \end{cases}$

$e^{-\frac{C}{2}} = e^{-\frac{(y_0-2)^2 - 2ln|1-t_0|}{2}} = |1-t_0|e^{-\frac{(y_0-2)^2}{2}}$

$\boxed{\text{Domain: } \begin{cases} t \geq 1 + |1-t_0|e^{-\frac{(y_0-2)^2}{2}} & \text{if } t_0 > 0 \\ t \leq 1 - |1-t_0|e^{-\frac{(y_0-2)^2}{2}} & \text{if } t_0 < 0. \end{cases}}$

Section 2.5:

Exponential Growth/Decay
Example: population growth/radioactive decay)

$y' = ry$, $y(0) = y_0$ implies $y = y_0 e^{rt}$

$r > 0$ $\qquad\qquad$ $r < 0$

Logistic growth: $y' = h(y)y$

Example: $y' = r(1 - \frac{y}{K})y$

y vs $f(y)$ $\qquad\qquad$ slope field:

Equilibrium solutions:

Asymptotically stable:

Asymptotically unstable:

Asymptotically semi-stable:

As $t \to \infty$, if $y > 0$, $y \to$ K
as seen from slope field

Solution: $y = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$

Linear vs Non-linear

linear: $a_0(t)y^{(n)} + ... + a_n(t)y = g(t)$

Determine if linear or non-linear:

Ex: $ty'' - t^3y' - 3y = sin(t)$

Ex: $2y'' - 3y' - 3y^2 = 0$

********Existence of a solution**************

********Uniqueness of solution***************

CH 2: Solve $\frac{dy}{dt} = f(t, y)$

2.2: Separation of variables: $N(y)dy = P(t)dt$

2.1: First order linear eqn: $\frac{dy}{dt} + p(t)y = g(t)$

Ex 1: $t^2y' + 2ty = tsin(t)$

Ex 2: $y' = ay + b$

Ex 3: $y' + 3t^2y = t^2$, $y(0) = 0$

Note: could use section 2.2 method, separation of variables to solve ex 2 and 3.

Ex 1: $t^2y' + 2ty = sin(t)$
(note, cannot use separation of variables).

$t^2y' + 2ty = sin(t)$

$(t^2y)' = sin(t)$

$\int (t^2y)'dt = \int sin(t)dt$

$(t^2y) = -cos(t) + C$ implies $y = -t^{-2}cos(t) + Ct^{-2}$

Gen ex: Solve $y' + p(x)y = g(x)$

Let $F(x)$ be an anti-derivative of $p(x)$

$e^{F(x)}y' + [p(x)e^{F(x)}]y = g(x)e^{F(x)}$

$e^{F(x)}y' + [F'(x)e^{F(x)}]y = g(x)e^{F(x)}$

$[e^{F(x)}y]' = g(x)e^{F(x)}$

$e^{F(x)}y = \int g(x)e^{F(x)}dx$

$y = e^{-F(x)} \int g(x)e^{F(x)}dx$

$y = e^{-F(x)}[A(x) + C]$

can uniquely solve for C is A+F exist

Find C given $y(t_0) = y_0$: $y_0 = 2 \pm \sqrt{2ln|1-t_0|+C}$

$\pm(y_0 - 2) = \sqrt{2ln|1-t_0|+C}$

$(y_0-2)^2 - 2ln|1-t_0| = C$

$y = 2 \pm \sqrt{2ln|1-t|+C}$

$y = 2 \pm \sqrt{2ln|1-t| + (y_0-2)^2 - 2ln|1-t_0|}$

$$\boxed{y = 2 \pm \sqrt{(y_0-2)^2 + ln\frac{(1-t)^2}{(1-t_0)^2}}}$$

Domain: $\begin{cases} t \geq e^{-\frac{C}{2}} + 1 & \text{if } t_0 > 0 \\ t \leq -e^{-\frac{C}{2}} + 1 & \text{if } t_0 < 0. \end{cases}$

$e^{-\frac{C}{2}} = e^{-\frac{(y_0-2)^2 - 2ln|1-t_0|}{2}} = |1-t_0|e^{-\frac{(y_0-2)^2}{2}}$

$$\boxed{\text{Domain: } \begin{cases} t \geq 1 + |1-t_0|e^{-\frac{(y_0-2)^2}{2}} & \text{if } t_0 > 0 \\ t \leq 1 - |1-t_0|e^{-\frac{(y_0-2)^2}{2}} & \text{if } t_0 < 0. \end{cases}}$$

Section 2.5: $y' = f(y)$

Exponential Growth/Decay
Example: population growth/radioactive decay)

$y' = ry$, $y(0) = y_0$ implies $y = y_0e^{rt}$

$r > 0$

exp growth

$r < 0$

exp decay

Logistic growth: $y' = h(y)y = f(y)$

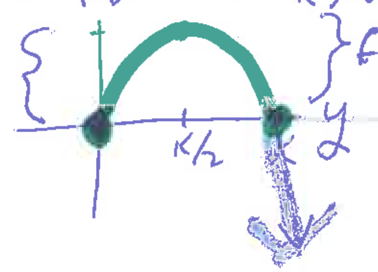
Example: $\boxed{y' = r(1 - \frac{y}{K})y}$

$f(y) = \text{slope} = r(1-\frac{y}{K})y$

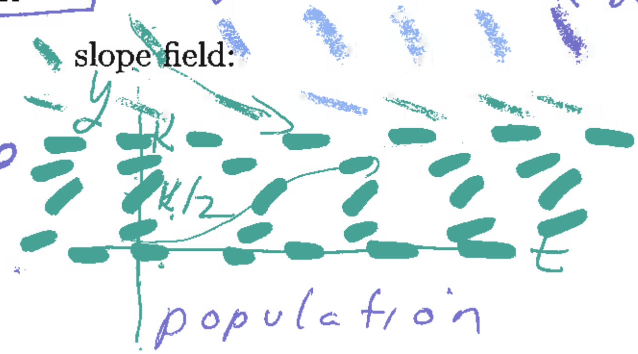
y vs $f(y)$

$f(y) = r(1-\frac{y}{K})y$

positive slope for y'



slope field:



population
Assume $y \geq 0$

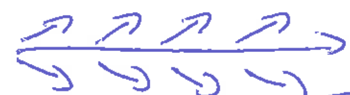
Equilibrium solutions:
$y = 0$, $y = K$

Asymptotically stable:

$y = K$

Asymptotically unstable:

$y = 0$



Asymptotically semi-stable:

none for this example

As $t \to \infty$, if $y > 0$, $y \to K$

Solution: $y = \frac{y_0K}{y_0+(K-y_0)e^{-rt}}$

2.3: Modeling with differential equations.

Ex.: $F = ma = mv'$

a = acceleration $= v' = x''$
v = velocity $= x'$
x = position
m = mass $\qquad mg$ = weight

Model 1: Falling ball near earth, neglect air resistance.

F_g = Gravitational force $= -mg$
IF the positive direction points up.
Note in some examples in the book, the positive direction points down ($F_g = +mg$) while in other examples in the book, the positive direction points up ($F_g = -mg$)

$mv' = -mg$ implies $v' = -g$. Thus $v = -gt + C$.

IVP: $v(0) = v_0$ implies $v_0 = -g(0) + C$ implies $C = v_0$. Thus $\boxed{v = -gt + v_0}$

$x' = v = -gt + v_0$ implies $x = -\frac{1}{2}gt^2 + v_0t + C$.

IVP: $x(0) = x_0$ implies $x_0 = -\frac{1}{2}g(0)^2 + v_0(0) + C$ implies $C = x_0$.

Thus $\boxed{x = -\frac{1}{2}gt^2 + v_0t + x_0.}$

$x = ?$

Note when ball reaches maximum height $v = 0$

Model 2: Falling ball near earth, include air resistance.

$F_g = -mg$

Let $A(v)$ = the force due to air resistance.

$mv' = F_g + A(v) = \boxed{-mg + A(v) = mv'}$

$A(v)$ (+)

F_g (−)

Model 3: Far from earth.

$F_g = -mg\frac{R^2}{(R+x)^2}$ where R = radius of the earth.

If x is small, $\frac{R^2}{(R+x)^2} \sim 1$ and thus $F_g = -mg$ when close to earth.

For large x, $mv' = -mg\frac{R^2}{(R+x)^2}$ where R constant.

$\frac{dv}{dt} = -mg\frac{R^2}{(R+x)^2}$ with 3 variables: v, t, x

To eliminate one variable: $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$

Note this trick can also be used to simplify some problems.

$$\frac{dx}{dx} \cdot \frac{dv}{dt} = \frac{dv}{dx}\left(\frac{dx}{dt}\right) = \frac{dv}{dx} v!$$