2.4 Solve Bernoulli's equation,

$$y' + p(t)y = g(t)y^n,$$

when $n \neq 0, 1$ by changing it

$$y^{-n}y' + p(t)y^{1-n} = g(t)$$

when $n \neq 0, 1$ by changing it to a linear equation by substituting $v = y^{1-n}$

Solve $ty' + 2t^{-2}y = 2t^{-2}y^5$

$ty^{-5}y' + 2t^{-2}y^{-4} = 2t^{-2}$

Let $v = y^{-4}$. Thus $v' = -4y^{-5}y'$

$-4ty^{-5}y' - 8t^{-2}y^{-4} = -8t^{-2}$

$tv' - 8t^{-2}v = -8t^{-2}$ ← linear ODE

Make coefficient of $v' = 1$

$v' - 8t^{-3}v = -8t^{-3}$

An antiderivative of $-8t^{-3}$ is $4t^{-2}$

Multiply equation by $e^{4t^{-2}}$

$e^{4t^{-2}}v' - 8t^{-3}e^{4t^{-2}}v = -8t^{-3}e^{4t^{-2}}$

$(e^{4t^{-2}}v)'$

$(e^{4t^{-2}}v)' = -8t^{-3}e^{4t^{-2}}$ by PRODUCT rule.

$\int (e^{4t^{-2}}v)'dt = -8\int t^{-3}e^{4t^{-2}}dt$

$e^{4t^{-2}}v = -8\int t^{-3}e^{4t^{-2}}dt.$

Let $u = 4t^{-2}$. Then $du = -8t^{-3}dt$

$e^{4t^{-2}}v = \int e^u du = e^u + C$

$e^{4t^{-2}}v = e^{4t^{-2}} + C$

$v = 1 + Ce^{-4t^{-2}}$

final answer in terms of y & t

$y^{-4} = 1 + Ce^{-4t^{-2}}$ implies $\boxed{y = \pm(1 + Ce^{-4t^{-2}})^{-\frac{1}{4}}}$

$y' + \frac{2}{t-3}y = 1$

An anti-derivative of $\frac{2}{t-3} = 2ln(t-3)$

$e^{2ln(t-3)} = e^{ln[(t-3)^2]} = (t-3)^2$

$y' + \frac{2}{t-3}y = 1$

$(t-3)^2y' + 2(t-3)y = (t-3)^2$

$\int [(t-3)^2y]' = \int (t-3)^2$

$(t-3)^2y = \frac{(t-3)^3}{3} + C$ implies $y = \frac{(t-3)}{3} + C(t-3)^{-2}$

Section 2.4 example: $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$

$F(y,t) = \frac{1}{(1-t)(2-y)}$ is continuous for all $t \neq 1$, $y \neq 2$

$$\frac{\partial F}{\partial y} = \frac{\partial\left(\frac{1}{(1-t)(2-y)}\right)}{\partial y} = \frac{1}{(1-t)}\frac{\partial(2-y)^{-1}}{\partial y} = \frac{1}{(1-t)(2-y)^2}$$

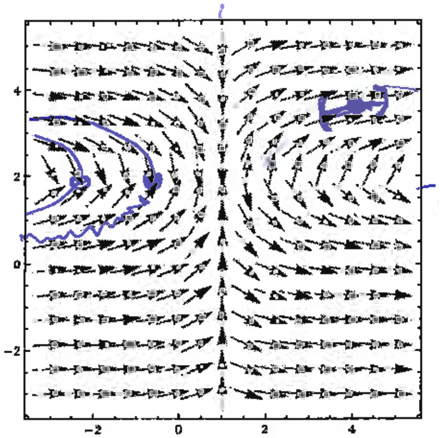
$\frac{\partial F}{\partial y}$ is continuous for all $t \neq 1$, $y \neq 2$

$$\boxed{\text{Thus the IVP } \frac{dy}{dt} = \frac{1}{(1-t)(2-y)},\ y(t_0) = y_0 \text{ has a unique solution if } t_0 \neq 1,\ y_0 \neq 2.}$$

Note that if $y_0 = 2$, $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$, $y(t_0) = 2$ has two solutions if $t_0 \neq \cancel{2}$ 1

Note that if $t_0 = 1$, $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$, $y(1) = y_0$ has no solutions.

t=1



$(1, 1/((1-x)(2-y)))/sqrt1 + 1/((1-x)(2-y))^2)$

Solve via separation of variables:

$\int(2-y)dy = \int \frac{dt}{1-t}$

$2y - \frac{y^2}{2} = -ln|1-t| + C$

$y^2 - 4y - 2ln|1-t| + C = 0$

$y = \frac{4 \pm \sqrt{16 + 4(2ln|1-t|+C)}}{2} = 2 \pm \sqrt{4 + 2ln|1-t| + C}$

$$\boxed{y = 2 \pm \sqrt{2ln|1-t| + C}}$$

Find domain: $2ln|1-t| + C \geq 0$

$2ln|1-t| \geq -C$

$ln|1-t| \geq -\frac{C}{2}$ Note: we want to find domain for this C and thus this C can't swallow constants).

$|1-t| \geq e^{-\frac{C}{2}}$ since e^x is an increasing function.

$1 - t \leq -e^{-\frac{C}{2}}$ or $1 - t \geq e^{-\frac{C}{2}}$

$-t \leq -e^{-\frac{C}{2}} - 1$ or $-t \geq e^{-\frac{C}{2}} - 1$

C's agree

$$\boxed{\text{Domain: } \begin{cases} t \geq e^{-\frac{C}{2}} + 1 & \text{if } t_0 > 0 \\ t \leq -e^{-\frac{C}{2}} + 1 & \text{if } t_0 < 0. \end{cases}}$$

Note: Domain is much easier to determine when the ODE is linear.

Find C given $y(t_0) = y_0$: $y_0 = 2 \pm \sqrt{2ln|1-t_0| + C}$

$\pm(y_0 - 2) = \sqrt{2ln|1-t_0| + C}$

$(y_0 - 2)^2 - 2ln|1-t_0| = C$

$y = 2 \pm \sqrt{2ln|1-t| + C}$

$y = 2 \pm \sqrt{2ln|1-t| + (y_0-2)^2 - 2ln|1-t_0|}$

$$\boxed{y = 2 \pm \sqrt{(y_0-2)^2 + ln\frac{(1-t)^2}{(1-t_0)^2}}}$$

Domain: $\begin{cases} t \geq e^{-\frac{C}{2}} + 1 & \text{if } t_0 > 0 \\ t \leq -e^{-\frac{C}{2}} + 1 & \text{if } t_0 < 0. \end{cases}$

$e^{-\frac{C}{2}} = e^{-\frac{(y_0-2)^2 - 2ln|1-t_0|}{2}} = |1-t_0|e^{-\frac{(y_0-2)^2}{2}}$

$$\boxed{\text{Domain: } \begin{cases} t \geq 1 + |1-t_0|e^{-\frac{(y_0-2)^2}{2}} & \text{if } t_0 > 0 \\ t \leq 1 - |1-t_0|e^{-\frac{(y_0-2)^2}{2}} & \text{if } t_0 < 0. \end{cases}}$$

Section 2.5: $y' = f(y)$

Exponential Growth/Decay
Example: population growth/radioactive decay)

$y' = ry$, $y(0) = y_0$ implies $y = y_0e^{rt}$

$r > 0$

exp growth

$r < 0$

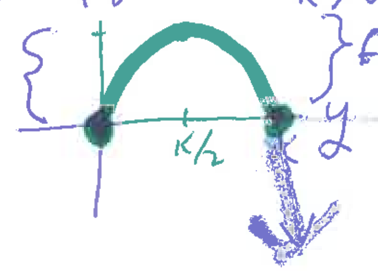
exp decay

Logistic growth: $y' = h(y)y = f(y)$

Example: $\boxed{y' = r(1 - \frac{y}{K})y}$ $f(y) = \text{slope} = r(1-\frac{y}{K})y$

y vs $f(y)$ $f(y) = r(1-\frac{y}{K})y$

positive slope for y'

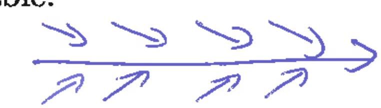


slope field:

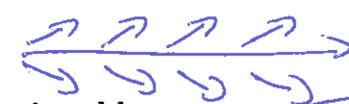
population
Assume $y \geq 0$

Equilibrium solutions:
$y = 0$, $y = K$

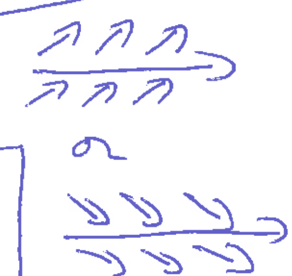
Asymptotically stable:
$y = K$



Asymptotically unstable:
$y = 0$



Asymptotically semi-stable:
none for this example



As $t \to \infty$, if $y > 0$, $y \to K$

Solution: $y = \frac{y_0K}{y_0 + (K-y_0)e^{-rt}}$

Linear vs Non-linear

linear: $a_0(t)y^{(n)} + ... + a_n(t)y = g(t)$

Determine if linear or non-linear:

Ex: $ty'' - t^3y' - 3y = sin(t)$

Ex: $2y'' - 3y' - 3y^2 = 0$

********Existence of a solution**************

********Uniqueness of solution***************

CH 2: Solve $\frac{dy}{dt} = f(t, y)$

2.2: Separation of variables: $N(y)dy = P(t)dt$

2.1: First order linear eqn: $\frac{dy}{dt} + p(t)y = g(t)$

Ex 1: $t^2y' + 2ty = tsin(t)$

Ex 2: $y' = ay + b$

Ex 3: $y' + 3t^2y = t^2$, $y(0) = 0$

Note: could use section 2.2 method, separation of variables to solve ex 2 and 3.

Ex 1: $t^2y' + 2ty = sin(t)$
(note, cannot use separation of variables).

$t^2y' + 2ty = sin(t)$

$(t^2y)' = sin(t)$

$\int (t^2y)'dt = \int sin(t)dt$

$(t^2y) = -cos(t) + C$ implies $y = -t^{-2}cos(t) + Ct^{-2}$

Gen ex: Solve $y' + p(x)y = g(x)$ ⇐ Linear ODE

Let $F(x)$ be an anti-derivative of $p(x)$

$e^{F(x)}y' + [p(x)e^{F(x)}]y = g(x)e^{F(x)}$

$e^{F(x)}y' + [F'(x)e^{F(x)}]y = g(x)e^{F(x)}$

$[e^{F(x)}y]' = g(x)e^{F(x)}$

$e^{F(x)}y = \int g(x)e^{F(x)}dx$

$y = e^{-F(x)} \int g(x)e^{F(x)}dx$

can determine domain base on $p(x)$ & $g(x)$

Section 2.7 Euler method: Using tangent lines to approximate a function.

$y_{i+1} = y_i + \Delta y = y_i + \frac{\Delta y}{\Delta t}\Delta t \cong y_i + \frac{dy}{dt}\Delta t$

Alternatively use equation of tangent line:

$$\text{slope} = \frac{y_{i+1}-y_i}{t_{i+1}-t_i} = f'(y_i, t_i).$$

$y_{i+1} = f'(y_i,t_i)(t_{i+1}-t_i)+y_i = T(t_{i+1})$ where $y = T(t)$ is the equation of the tangent line at (y_i, t_i).

Example: $\frac{dy}{dt} = y^2$, $y(2) = 1$ implies $y = \frac{1}{3-t}$.

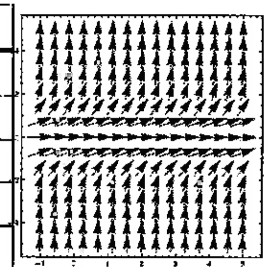
t	$y = 1/(3-t)$	approximation
2.000000	1.000000	1.000000
3.000000	~~999.000000~~ ∞	2.000000
4.000000	-1.000000	6.000000
5.000000	-0.500000	42.000000
6.000000	-0.333333	1806.000000

t	$y = 1/(3-t)$	approximation
2.000000	1.000000	1.000000
2.100000	1.111111	1.100000
2.200000	1.250000	1.221000
2.300000	1.428571	1.370084
2.400000	1.666667	1.557797
2.500000	2.000000	1.800470
2.600000	2.500000	2.124640
2.700000	3.333333	2.576049
2.800000	5.000000	3.239652
2.900000	10.000004	4.289186

$\Delta t = 0.1$

t	$y = 1/(3-t)$	approximation
2.00	1.000000	1.000000
2.01	1.010101	1.010000
2.02	1.020408	1.020201
2.03	1.030928	1.030609
2.04	1.041667	1.041231
2.05	1.052632	1.052072
2.06	1.063830	1.063141
2.07	1.075269	1.074443
2.08	1.086957	1.085988
2.09	1.098901	1.097782
2.10	1.111111	1.109833
2.11	1.123595	1.122150
2.12	1.136364	1.134742
2.13	1.149425	1.147619
2.87	7.692308	6.721314
2.88	8.333333	7.173075
2.89	9.090908	7.687605
2.90	9.999998	8.278598
2.91	11.111107	8.963949
2.92	12.499993	9.767473
2.93	14.285716	10.721509
2.94	16.666666	11.871017
2.95	19.999996	13.280227
2.96	24.999987	15.043871
2.97	33.333298	17.307051
2.98	49.999897	20.302391
2.99	99.999496	24.424261

$\Delta t = 0.01$



$y' = y^2$