22M:100 (MATH:3600:0001) Final Exam
May 15, 2013 SHOW ALL STEPS

[13] 1.) Solve: $ty' + 6t^2y - t^2 = 0$, $y(1) = 3$

$y' + 6ty = t$, $\int 6tdt = 3t^2$

$e^{3t^2}y' + 6te^{3t^2} = te^{3t^2}$ implies $[e^{3t^2}y]' = te^{3t^2}$ implies $\int [e^{3t^2}y]'dt = \int te^{3t^2}dt$

implies $e^{3t^2}y = \frac{e^{3t^2}}{6} + C$ (via u-substitution). Thus $y = \frac{1}{6} + Ce^{-3t^2}$

$y(1) = 3$: $3 = \frac{1}{6} + Ce^{-3}$ implies $C = \frac{17}{6}e^3$

Answer: $y = \frac{1}{6} + \frac{17}{6}e^{-3t^2+3}$

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[13] 2.) Solve: $(2x^3y^2 - 3x)dx + (x^4y - y^{-1})dy = 0$

Note: $\frac{\partial}{\partial y}(2x^3y^2 - 3x) = 4x^3y = \frac{\partial}{\partial x}(x^4y - y^{-1})$. Thus equation is exact.

Let $\psi_x = 2x^3y^2 - 3x$. Then $\psi = \frac{x^4y^2 - 3x^2}{2} + h(y)$ and $\psi_y = x^4y + h'(y) = x^4y - y^{-1}$

$\int h'(y)dy = \int -y^{-1}dy = -ln|y|$ and $\frac{x^4y^2-3x^2}{2} - ln|y| = C$

Answer: $\frac{x^4y^2-3x^2}{2} - ln|y| = C$

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[14] 3.) Use the infinite series solution method (ch 5) about the point $x_0 = 0$ to solve the differential equation, $(x + 1)y' - 2y = 0$.

Suppose $y = \sum_{n=0}^{\infty} a_n x^n$. Then $y' = \sum_{n=1}^{\infty} na_n x^{n-1}$

$(x + 1)\sum_{n=1}^{\infty} na_n x^{n-1} - 2\sum_{n=0}^{\infty} a_n x^n = 0.$

$\sum_{n=1}^{\infty}(x + 1)na_n x^{n-1} - 2\sum_{n=0}^{\infty} a_n x^n = 0.$

$\sum_{n=1}^{\infty} na_n x^n + \sum_{n=1}^{\infty} na_n x^{n-1} - 2\sum_{n=0}^{\infty} a_n x^n = 0.$

$\sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty}(n + 1)a_{n+1}x^n - 2\sum_{n=0}^{\infty} a_n x^n = 0.$

$\sum_{n=1}^{\infty} na_n x^n + \sum_{n=1}^{\infty}(n + 1)a_{n+1}x^n - 2\sum_{n=1}^{\infty} a_n x^n + a_1 - 2a_0 = 0.$

$\sum_{n=1}^{\infty}[na_n + (n + 1)a_{n+1} - 2a_n]x^n + a_1 - 2a_0 = 0.$

hence $a_1 - 2a_0 = 0$ and $na_n + (n + 1)a_{n+1} - 2a_n = 0$ for all $n \geq 1$

hence $a_1 = 2a_0$ and $a_{n+1} = \frac{(2-n)a_n}{n+1}$ for all $n \geq 1$

Thus $a_2 = \frac{(2-1)a_1}{2} = \frac{a_1}{2} = \frac{2a_0}{2} = a_0$. And $a_3 = \frac{(2-2)a_2}{2+1} = 0$. Thus $a_4 = \frac{(2-3)a_3}{3+1} = 0$. Thus $a_n = 0\ \forall n \geq 3$.

Thus $y = a_0x^2 + 2a_0x + a_0$

Check: $y' = 2a_0x + 2a_0$

$(x + 1)(2a_0x + 2a_0) - 2(a_0x^2 + 2a_0x + a_0). = 2(a_0x^2 + a_0x + a_0x + a_0 - a_0x^2 - 2a_0x - a_0) = 0.$

Answer: $y = a_0x^2 + 2a_0x + a_0$

[20] 4.) Solve: $y'' - 3y' - 4y = 8t$, $y(0) = 5$, $y'(0) = 3$

$r^2 - 3r - 4 = 0.$ $(r-4)(r+1) = 0.$ Thus $r = 4, -1.$

general homogeneous solution: $y = c_1 e^{4t} + c_2 e^{-t}$

Let $y = At + B$. Then $y' = A$ and $y'' = 0$

$-3A - 4(At + B) = 8t$

$A = -2$, $6 - 4B = 0$. Thus $B = \frac{3}{2}$

Thus $\psi = -2t + \frac{3}{2}$ is a non-homogeneous solution.

General solution to non-homogeneous DE: $y = c_1 e^{4t} + c_2 e^{-t} - 2t + \frac{3}{2}$

$y' = 4c_1 e^{4t} - c_2 e^{-t} - 2$

$5 = c_1 + c_2 + \frac{3}{2}$

$3 = 4c_1 - c_2 - 2$

Thus $c_1 = \frac{17}{10}$ and $c_2 = \frac{9}{5}$

Answer: $y = \frac{17}{10}e^{4t} + \frac{9}{5}e^{-t} - 2t + \frac{3}{2}$

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[20] 5.) Suppose $\frac{dx}{dt} = (x-3)(y+2)$ and $\frac{dy}{dt} = 2x - y$.

(a.) State all critical points of this system of differential equations (3, 6), (-1, -2)

$(x-3)(y+2) = 0$ implies $x = 3$ or $y = -2$.

$2x - y = 0$ implies if $x = 3$, then $y = 6$ and if $y = -2$, then $x = -1$.

(b.) State two solutions to this system of differential equations.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} \text{ and } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

(c.) For the critical point with negative x coordinate, find the corresponding linear system.

$\begin{pmatrix} y+2 & x-3 \\ 2 & -1 \end{pmatrix}$. For $(x, y) = (-1, -2)$, the Jacobian matrix is $\begin{pmatrix} 0 & -4 \\ 2 & -1 \end{pmatrix}$.

Thus $\mathbf{x}' = \begin{pmatrix} 0 & -4 \\ 2 & -1 \end{pmatrix} \mathbf{x}$ is the corresponding linear system where the critical point (-1, -2) has been translated to the origin.

(d.) Based on the linear translated approximation found in part b, what conclusions regarding stability can you draw about the nonlinear system near the critical point with negative x coordinate?

$(-\lambda)(-1-\lambda) + 8 = (\lambda + \lambda^2) + 8 = \lambda^2 + \lambda + 8 = 0$. Thus the Jacobian matrix at $(x, y) = (-1, -2)$ has two complex eigenvalues $\lambda = \frac{-1 \pm \sqrt{1-32}}{2}$ that are neither real nor imaginary where the real part is negative.

Hence the (-1, -2) is an asymtotically stable spiral source.