Math 100 Differential Equations Exam #1
February 27, 2013 SHOW ALL WORK

[10] 1.) By giving a specific example, prove that $f : R \to R,\ f(x) = e^x$ is not onto.

Answer 1: We know that $e^x > 0$ for all x. Thus there is no x such that $f(x) = e^x = -1$. Hence -1 is not in the image of f.

Answer 2: Suppose $f(x) = -1$. Then $e^x = -1$. But then $x = ln(-1)$. But $ln(-1)$ is not defined. Thus -1 is not in the image of f.

2.) Circle T for true and F for false. Note that the answer to 2a is true.

[3] 2a.) In more advanced math classes, you may be required to provide many more details when proving a function is onto.

T

[4] 2b.) Suppose ϕ is a solution to the equation, $y' + p(t)y = g(t)$, then 2ϕ must also be a solution to $y' + p(t)y = g(t)$.

F

[4] 2c.) Suppose ϕ is a solution to the equation, $y' + p(t)y^2 = 0$, then 2ϕ must also be a solution to $y' + p(t)y^2 = 0$.

F

[4] 2d.) Suppose ϕ is a solution to the equation, $y' + p(t)y = 0$, then 2ϕ must also be a solution to $y' + p(t)y = 0$.

T

[15] 3.) Draw the direction field for $y' = \frac{1}{2}y + 1$. Determine if there are any equilibrium solutions. If so, determine if the equilibrium solution(s) are stable, unstable or semi-stable.

Equilibrium solution = constant solution. Thus $y' = 0$.

$\frac{1}{2}y + 1 = 0$ implies $\frac{1}{2}y = -1$. Thus $y = -2$ is the equilibrium solution.

[15] 4.) Solve the following initial value problem: $y'y = t + 3ty^2$, $y(0) = -2$

$y'y = t + 3ty^2$

$\frac{dy}{dt}y = t(1 + 3y^2)$

$\frac{ydy}{1+3y^2} = tdt$

$\int \frac{ydy}{1+3y^2} = \int tdt$

$\frac{1}{6}\int \frac{6ydy}{1+3y^2} = \int tdt$

$\frac{1}{6}ln|1 + 3y^2| = \frac{1}{2}t^2 + C$

$ln|1 + 3y^2| = 3t^2 + C$

$e^{ln|1+3y^2|} = e^{3t^2+C}$

$|1 + 3y^2| = e^{3t^2}e^C$

$1 + 3y^2 = \pm e^C e^{3t^2}$

$1 + 3y^2 = Ce^{3t^2}$

$3y^2 = Ce^{3t^2} - 1$

$y^2 = \frac{Ce^{3t^2}-1}{3}$

$y = \pm\sqrt{\frac{Ce^{3t^2}-1}{3}}$

$y(0) = -2$: $\quad -2 = -\sqrt{\frac{Ce^0-1}{3}} = -\sqrt{\frac{C-1}{3}}$

$4 = \frac{C-1}{3}$ implies $12 = C - 1$ implies $C = 13$.

Answer 4.) $y = -\sqrt{\frac{13e^{3t^2}-1}{3}}$

5.) Find the general solutions for the following three differential equations.

[15] 5A.) $2y'' - 3y' + 5y = 0$

$y = e^{rt}$. Then $y' = re^{rt}$ and $y'' = r^2e^{rt}$.

$2r^2e^{rt} - 3re^{rt} + 5e^{rt} = 0$ implies $2r^2 - 3r + 5 = 0$

$2r^2 - 3r + 5 = 0$ implies $r = \frac{3 \pm \sqrt{9-4(2)(5)}}{4} = \frac{3 \pm \sqrt{9-40}}{4} = \frac{3 \pm \sqrt{-31}}{4} = \frac{3 \pm \sqrt{-31}}{4} = \frac{3}{4} \pm i\frac{\sqrt{31}}{4}$

Answer 5A.) $\underline{\;y = c_1 e^{\frac{3t}{4}} cos(\frac{\sqrt{31}}{4}\; t) + c_2 e^{\frac{3t}{4}} sin(\frac{\sqrt{31}}{4}\; t)\;}$

[15] 5B.) $y'' + 6y' + 9y = 0$

$r^2 + 6r + 9 = 0$

$(r+3)^2 = 0$ implies $r = -3$

Answer 5B) $\underline{\;y = c_1 e^{-3t} + c_2 te^{-3t}\;}$

[15] 5C.) $3y''(y')^2 = 1$

Let vy'. Then $v' = y''$.

$3v'(v)^2 = 1$ implies $3\frac{dv}{dt}(v)^2 = 1$

$3dv(v)^2 = dt$

$\int 3(v)^2 dv = \int dt$

$v^3 = t + C_1$. Thus $v = (t + C_1)^{\frac{1}{3}}$

$\frac{dy}{dt} = (t + C_1)^{\frac{1}{3}}$

$dy = (t + C_1)^{\frac{1}{3}} dt$

$\int dy = \int (t + C_1)^{\frac{1}{3}} dt$

$y = \frac{3}{4}(t + C_1)^{\frac{4}{3}} + C_2$

Answer 5C.) $\underline{\;y = \frac{3}{4}(t + C_1)^{\frac{4}{3}} + C_2\;}$