$\mathcal{L}$ is a linear function.

Ex: $\mathcal{L}(5e^2 e^{it} e^{3t}) =$

Solve (i.e., solve for y):

$$y^{iv} = 0, \quad y(0) = 0,\ y'(0) = 0,\ y''(0) = 0,\ y'''(0) = 10$$

$$\mathcal{L}(y^{iv}) = \mathcal{L}(0)$$

$$s^4\mathcal{L}(y) - s^3 10y(0) - s^2y'(0) - sy''(0) - y'''(0) = 0$$

$$s^4\mathcal{L}(y) - 10 = 0$$

$$s^4\mathcal{L}(y) = 10$$

$$\mathcal{L}(y) = \frac{10}{s^4}$$

$$\mathcal{L}^{-1}(\mathcal{L}(y)) = \mathcal{L}^{-1}(\frac{10}{s^4})$$

$y = \mathcal{L}^{-1}(\frac{10}{s^4})$ Formula 3: $\mathcal{L}^{-1}(\frac{n!}{s^{n+1}}) = t^n$

$$y = \mathcal{L}^{-1}(\frac{10}{3!} \cdot \frac{3!}{s^4}) = \frac{10}{3!}\mathcal{L}^{-1}(\cdot \frac{3!}{s^4}) = \frac{10}{3!}t^3 = \frac{5}{3}t^3$$

Solve (i.e., solve for y):

$$y'' + 2y' + 5y = 0, \quad y(0) = 1,\ y'(0) = 7$$

$$\mathcal{L}(y'' + 2y' + 5y) = \mathcal{L}(0)$$

$$\mathcal{L}(y'') + 2\mathcal{L}(y') + 5\mathcal{L}(y) = 0$$

$$s^2\mathcal{L}(y) - sy(0) - y'(0)$$
$$+2[s\mathcal{L}(y) - y(0)]$$
$$+5\mathcal{L}(y) = 0$$

$$s^2\mathcal{L}(y) - s - 7$$
$$+2[s\mathcal{L}(y) - 1]$$
$$+5\mathcal{L}(y) = 0$$

$$s^2\mathcal{L}(y) - s - 7$$
$$+2s\mathcal{L}(y) - 2$$
$$+5\mathcal{L}(y) = 0$$

$$(s^2 + 2s + 5)\mathcal{L}(y) = s + 7 + 2$$

$$\mathcal{L}(y) = \tfrac{s+9}{s^2+2s+5}$$

$$\mathcal{L}^{-1}(\mathcal{L}(y)) = \mathcal{L}^{-1}(\tfrac{s+9}{s^2+2s+5})$$

$$y = \mathcal{L}^{-1}(\tfrac{s+9}{s^2+2s+5})$$

$y = \mathcal{L}^{-1}(\frac{s+9}{s^2+2s+1-1+5})$

$y = \mathcal{L}^{-1}(\frac{s+9}{(s+1)^2-1+5})$

$y = \mathcal{L}^{-1}(\frac{s+9}{(s+1)^2+4})$

$y = \mathcal{L}^{-1}(\frac{s+1+8}{(s+1)^2+4})$

$y = \mathcal{L}^{-1}(\frac{s+1}{(s+1)^2+4} + \frac{8}{(s+1)^2+4})$

$y = \mathcal{L}^{-1}(\frac{s+1}{(s+1)^2+4}) + \mathcal{L}^{-1}(\frac{8}{(s+1)^2+4})$

$y = \mathcal{L}^{-1}(\frac{s+1}{(s+1)^2+4}) + 4\mathcal{L}^{-1}(\frac{2}{(s+1)^2+4})$

$y = e^{-t}cos(2t) + 4e^{-t}sin(2t)$

Partial Check: Plug $y = e^{rt}$ into $y'' + 2y' + 5y = 0$:

Characteristic eqn: $r^2 + 2r + 5 = 0$ implies

$$(r+1)^2 + 4 = 0 \text{ and thus } (r+1)^2 = -4.$$

Hence $r + 1 = \sqrt{-4} = \pm 2i$. Thus $r = -1 \pm 2i$

and the general homogeneous soln is

$$y = c_1 e^{-t}cos(2t) + c_2 e^{-t}sin(2t)$$

Partial Check:

$y(0) = 1$ for IVP soln $y = e^{-t}cos(2t) + 4e^{-t}sin(2t)$

$1 = e^0 cos(0) + 4e^0 sin(0)$ where equality does hold

Partial Check: $y'(0) = 7$ - too much work due to product rule.