4.4 = 3.6 Variation of Parameters

$$\text{Solve } y''' - 4y'' + y' + 6y = 8e^{4t}$$

Step 1: Find homogeneous solutions:

Solve $y''' - 4y'' + y' + 6y = 0$

Plug in $y = e^{rt}$ and simplify:

$r^3 - 4r^2 + r + 6 = (r+1)(r^2 - 5r + 6) = (r+1)(r-2)(r-3) = 0$

Thus $r = -1, 2, 3$ and general homogeneous solution is

$$y = c_1 e^{-t} + c_2 e^{2t} + c_3 e^{3t}$$

Step 2: Find one non-homogeneous solution. Since the general solution to non-homogeneous DE is

$$y = c_1 e^{-t} + c_2 e^{2t} + c_3 e^{3t} + \psi$$

3.5 = 4.3: where by the undetermined coefficients method,

$$\psi = Ae^{4t}$$

for some constant A that can be determined by plugging ψ into the DE and solving for A.

Note 1: $y = e^{4t}$ is NOT a homogeneous solution, so it is a good guess.

Note 2: Undetermined coefficients method is usually much much easier than variation of parameters.

But since the undetermined coefficients method cannot always be used (sometimes guess is impossible to determine), we will instead use the much longer method of variation of parameters to illustrate this method.

Thus by **3.6 = 4.4**, variation of parameters method,

$$\psi = u_1 e^{-t} + u_2 e^{2t} + u_3 e^{3t}$$

where u_i are unknown functions that we can determine via the following:

If homogeneous solution is $y = c_1\phi_1 + c_2\phi_2 + c_3\phi_3$, then a non-homogeneous solution is

$$y = u_1(t)\phi_1 + u_2(t)\phi_2 + u_3(t)\phi_3$$

where we can determine the unknown functions by plugging this into the DE. But we have 3 unknowns. Thus we might want 3 equations to solve for the unknowns. Thus we can choose 2 equations to simplify the algebra.

We need to calculate y' which by the product rule will have 6 terms. Thus we will set the sum of 3 of these terms to 0 to avoid 2nd derivative of u_i when calculating y''.

Thus we choose $u_1'(t)\phi_1 + u_2'(t)\phi_2 + u_3'(t)\phi_3 = 0$

Similarly to avoid 2nd derivative of u_i when calculating y''', we choose $u_1'(t)\phi_1' + u_2'(t)\phi_2' + u_3'(t)\phi_3' = 0$.

We plug our potential solution into the DE and after lots of simplification, we obtain the 3rd equation:

$$u_1'(t)\phi_1'' + u_2'(t)\phi_2'' + u_3'(t)\phi_3'' = \frac{g(t)}{a}$$

Thus to summarize, to find u_i, we solve the following system of DE (via Cramer's rule) and then integrate

$$\begin{gathered} u_1'(t)\phi_1 + u_2'(t)\phi_2 + u_3'(t)\phi_3 = 0 \\ u_1'(t)\phi_1' + u_2'(t)\phi_2' + u_3'(t)\phi_3' = 0 \\ u_1'(t)\phi_1'' + u_2'(t)\phi_2'' + u_3'(t)\phi_3'' = \tfrac{g(t)}{a} \end{gathered}$$

Or in matrix form:

$$\begin{pmatrix} \phi_1 & \phi_2 & \phi_3 \\ \phi_1' & \phi_2' & \phi_3' \\ \phi_1'' & \phi_2'' & \phi_3'' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{g(t)}{a} \end{pmatrix}$$

Thus for our example: $\mathbf{1}y''' - 4y'' + y' + 6y = 8e^{4t}$,

where the homogeneous solution is $y = c_1e^{-t} + c_2e^{2t} + c_3e^{3t}$,

and if we use the much longer method of variation of parameters (3.6 = 4.4), our non-homogeneous solution is

$$y = u_1(t)e^{-t} + u_2(t)e^{2t} + u_3(t)e^{3t}$$

where we can solve for the unknown functions u_i by solving for the unknowns u_i' in the following system of equations and then integrating.

$$\begin{gathered} u_1'(t)e^{-t} + u_2'(t)e^{2t} + u_3'(t)e^{3t} = 0 \\ u_1'(t)(-e^{-t}) + u_2'(t)(2e^{2t}) + u_3'(t)(3e^{3t}) = 0 \\ u_1'(t)e^{-t} + u_2'(t)(4e^{2t}) + u_3'(t)(9e^{3t}) = 8e^{4t} \end{gathered}$$

Or in matrix form:

$$\begin{pmatrix} e^{-t} & e^{2t} & e^{3t} \\ -e^{-t} & 2e^{2t} & 3e^{3t} \\ e^{-t} & 4e^{2t} & 9e^{3t} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 8e^{4t} \end{pmatrix}$$

Note the coefficient matrix is the matrix whose determinant is the Wronskian!!!!!

We can solve for the unknowns u_1', u_2',u_3' using Cramer's rule.

To calculate Wronskian:

Method 1: Use Definition.

Method 2: Use Abel's Theorem and $W(t_0)$.

See class notes for rest of solution.

Linear Algebra Review: Determinants

Defn: $detA = \Sigma \pm a_{1j_1}a_{2j_2}...a_{nj_n}$

2×2 short-cut: $det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{21}a_{12}$

3×3 short-cut: $det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$

Note there is no short-cut for $n \times n$ matrices when $n > 3$.

Definition of Determinant using cofactor expansion

Defn: A_{ij} is the matrix obtained from A by deleting the ith row and the jth column.

Thm: Let $A = (a_{ij})$ by an $n \times n$ square matrix, $n > 1$. Then expanding along row i,

$$detA = \Sigma_{k=1}^{n}(-1)^{i+k}a_{ik}detA_{ik}.$$

Or expanding along column j,

$$detA = \Sigma_{k=1}^{n}(-1)^{k+j}a_{kj}detA_{kj}.$$

Properties of Determinants

Thm: If $A \xrightarrow[R_i \to cR_i]{} B$, then $detB = c(detA)$.

Thm: If $A \xrightarrow[R_i \leftrightarrow R_j]{} B$, then $detB = -(detA)$.

Thm: If $A \xrightarrow[R_i + cR_j \to \quad R_i]{} B$, then $detB = detA$.

Some Shortcuts:

Thm: If A is an $n \times n$ matrix which is either lower triangular or upper triangular, then $detA = a_{11}a_{22}...a_{nn}$, the product of the entries along the main diagonal.

Thm: A square matrix is invertible if and only if $detA \neq 0$.