Guess a possible non-homog soln for the following DEs:

Note homogeneous solution to $y'' - 4y' - 5y = 0$ is $y = c_1e^{-t} + c_2e^{5t}$

$$\text{since } r^2 - 4r - 5 = (r-5)(r+1) = 0$$

1.) $y'' - 4y' - 5y = 4e^{2t}$

Guess: $\underline{\quad y \;=\; Ae^{2t} \quad}$

We want to plug in a solution for y into the left-hand side (LHS) of the equation that will give us the right-hand side (RHS) of the equation. In this case we need the output of the LHS to be a multiple of e^{2t} (in particular the output of the LHS when pluggin in y needs to be the RHS which is $4e^{2t}$).

Thus we guess $y = Ae^{2t}$. Plugging this into the LHS, we can solve for A so that we get the RHS, $4e^{2t}$, thus finding a non-homogeneous solution.

2a.) $y'' - 4y' - 5y = t^2 - 2t + 1$

Guess: $\underline{\quad y \;=\; At^2 + Bt + C \quad}$

2b.) $y'' - 4y' - 5y = t^2$

Guess: $\underline{\quad y \;=\; At^2 + Bt + C \quad}$

2c.) $y'' - 4y' - 5y =$ a degree 2 polynomial

Guess: $\underline{\quad y \;=\; At^2 + Bt + C \quad}$

Note that the non-homog solution guess is the same for 2a, 2b, 2c. In each case, we need to guess a solution such that when we plug it into the LHS we get the RHS, a degree 2 polynomial. Thus our guess is a degree 2 polynomial (but compare this to example 12). Note we need $y = At^2 + Bt + C$ even in case 2b. Make sure you understand why $y = At^2$ won't work.

3a.) $y'' - 4y' - 5y = 30$

Guess: $\underline{\quad y \;=\; A \quad}$

We have a constant on the RHS, so I guess a constant (but again compare to example 12). If you are observant, you may note that a non-homog solution is $y = -6$)

4a.) $y'' - 4y' - 5y = 4sin(3t)$

Guess: $\underline{\quad y \;=\; Asin(3t) + Bcos(3t) \quad}$

4b.) $y'' - 4y' - 5y = 4sin(3t) + 5cos(3t)$

Guess: $\underline{\quad y \;=\; Asin(3t) + Bcos(3t) \quad}$

4c.) $y'' - 4y' - 5y = 5cos(3t)$

Guess: $\underline{\quad y \;=\; Asin(3t) + Bcos(3t) \quad}$

Note that the non-homog solution guess is the same for 4a, 4b, 4c. If we plug in $y = Asin(3t)$, the output will contain both $sin(3t)$ and $cos(3t)$ terms. Thus I need to include both these terms in my guess. Compare to example 11.

5.) $y'' - 4y' - 5y = 4e^{-t}$

Guess: $\underline{\quad y \;=\; Ate^{-t} \quad}$

Note $y = Ae^{-t}$ is a homogeneous solution. Thus if I plug in it, I will get 0. But I want the RHS, $4e^{-t}$. When a guess doesn't work because it is a homogeneous solution, multiple by t.

Sidenote: this trick works because when you plug it in, you must use the product rule; the homogeneous part e^{-t} of $y = Ate^{-t}$ will result in a number of cancellations, but the t part will give you terms that don't cancel out and whose sum is the RHS.

Observe $y = Ate^{-t}$, $\quad y' = Ae^{-t} - Ate^{-t}$, $\quad y'' = -Ae^{-t} - Ae^{-t} + Ate^{-t} = -2Ae^{-t} + Ate^{-t}$

Thus $y'' - 4y' - 5y = -2Ae^{-t} + Ate^{-t} - 4(Ae^{-t} - Ate^{-t}) - 5(Ate^{-t})$

$= -2Ae^{-t} - 4Ae^{-t} + At(e^{-t} + 4e^{-t} - 5e^{-t}) = -6Ae^{-t} + At(0) = 4e^{-t}$ when $A = -\frac{2}{3}$

DO NOT FORGET THE PRODUCT RULE!!!!

6.) $y'' - 4y' - 5y = (e^t) + (e^{-t}) + (2t^3 + 3t^2) + (4sin(3t) + 5cos(3t))$

Guess: $y = (A_1e^t) + (A_2te^{-t}) + (A_3t^3 + B_3t^2 + C_3t + D_3) + (A_4sin(3t) + B_4cos(3t))$

Note if we wanted to find a non-homogeneous solution, we would need to determine all our undetermined coefficients. Note we have 8 undetermined coefficients. Instead of solving for them all at once (which would require 8 equations for the 8 unknowns), it is easier to divide finding a non-homogeneous solution into 4 simpler parts indicated by the parenthesis and subscripts as described below:

a.) Find A_1 by plugging $y = A_1e^t$ into $y'' - 4y' - 5y = e^t$

b.) Find A_2 by plugging $y = A_2te^{-t}$ into $y'' - 4y' - 5y = e^{-t}$

c.) Find A_3, B_3, C_3, D_3 by plugging $y = A_3t^3 + B_3t^2 + C_3t + D_3$ into $y'' - 4y' - 5y = 2t^3 + 3t^2$

d.) Find A_4, B_4 by plugging $y = A_4sin(3t) + B_4cos(3t)$ into
$y'' - 4y' - 5y = 4sin(3t) + 5cos(t)$

We get the non-homogeneous solution by adding together the non-homogeneous solutions obtained from the above 4 parts since our diff eqn is LINEAR.

We get the general solution by combining the general homogeneous solution with this non-homogeneous solution.

7.) $y'' - 4y' - 5y = e^t + e^{-t} + 2t^3 + 3t^2 + 4sin(3t) + 5cos(t)$

Guess: $y = (A_1e^t) + (A_2te^{-t}) + (A_3t^3 + B_3t^2 + C_3t + D_3)$

$+(A_4sin(3t) + B_4cos(3t)) + (A_5sin(t) + B_5cos(t))$

8.) $y'' - 4y' - 5y = 4(t^2 - 2t - 1)e^{2t}$

Guess: $y = (At^2 + Bt + C)e^{2t}$

Since the RHS is a product, we guess a product.
Note I could have guessed $y = (At^2 + Bt + C)De^{2t} = (ADt^2 + BDt + CD)e^{2t}$, but since AD, BD, CD are just constants, I don't need D.

Note homogeneous solution to $y'' - 6y' + 9y = 0$ is $y = c_1 e^{3t} + c_2 te^{3t}$

since $r^2 - 6r + 9 = (r-3)(r-3) = 0$

9.) $y'' - 6y' + 9y = 7e^{3t}$

Guess: $\underline{\quad y \ = \ At^2 e^{3t} \quad}$

Note neither $y = Ae^{3t}$ nor $y = Ate^{3t}$ will work since both are homogeneous solutions. But our trick of multiplying by t until we have a guess that is not a homogeneous solution will work.

10.) $y'' - 6y' + 9y = 7e^{-3t}$

Guess: $\underline{\quad y \ = \ Ae^{-3t} \quad}$

$y \ = \ Ae^{-3t}$ is not a homogeneous solution (when $A \neq 0$).

Some special cases:

11.) $y'' - 5y = 4sin(3t)$

Best Guess: $\underline{\quad y \ = \ Asin(3t) \quad}$

Note, we also could have guessed $y = Asin(3t) + Bcos(3t)$, but since there is no y' term, we don't need the cosine term. But both guesses will work. Plugging in $y = Asin(3t) + Bcos(3t)$ will take a little more work, but you will still get the right answer.

12.) $y'' - 4y' = t^2 - 2t + 1$

Guess: $\underline{\quad y \ = \ At^3 + Bt^2 + Ct \quad}$

Note there is no y term on the LHS. Thus to get a t^2 term when we plug in our guess, we will need to plug in a t^3 term. Hence we guess a degree 3 polynomial. Note we don't need to include a constant term; we could have guessed $y \ = \ At^3 + Bt^2 + Ct + D$, but any constant D will work (and hence there are an infinite number of solutions for D) so we might as well take $D = 0$.

Don't worry too much about guessing wrong. You will usually be able to figure out why an incorrect guess doesn't work and use that info to determine a better guess.