A Quick Review of
Linear Algebra
(linear combination, linear independence, span, basis)

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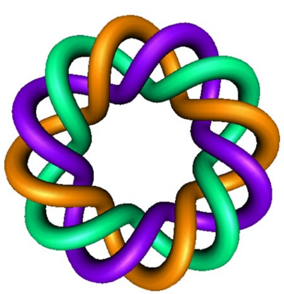
Partial Fractions
for
Differential Equations



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LINEAR COMBINATION

$\mathbf{p}$ is a linear combination of $\{\mathbf{b_1}, \mathbf{b_2}, \cdots, \mathbf{b_n}\}$ iff there exists c_i such that

$c_i \in \mathbb{R}$

$$\mathbf{p} = c_1\mathbf{b_1} + c_2\mathbf{b_2} + \ldots + c_n\mathbf{b_n}$$

Example 1:
Let $\mathbf{b_1} = (1,0,0)$, $\mathbf{b_2} = (0,1,0)$, $\mathbf{b_3} = (0,0,1)$.

$(1,2,3)$ is linear combination of

$$\{(1,0,0),(0,1,0),(0,0,1)\}$$

since $(1,2,3) = 1((1,0,0)) + 2((0,1,0)) + 3((0,0,1))$

LINEAR COMBINATION

$\mathbf{p}$ is a linear combination of $\{\mathbf{b_1}, \mathbf{b_2}, \cdots, \mathbf{b_n}\}$ iff there exists c_i such that

$$\mathbf{p} = c_1\mathbf{b_1} + c_2\mathbf{b_2} + \ldots + c_n\mathbf{b_n}$$

Example 2: Let $\mathbf{b_1} = 1$, $\mathbf{b_2} = t$, $\mathbf{b_3} = t^2$

Then $1 + 2t + 3t^2$ is a linear combination of $\{1, t, t^2\}$

Sidenote: $(1, 2, 3)$ can be used to represent the polynomial $1 + 2t + 3t^2$.

Sidenote = we won't need this for this class.

EXISTENCE

$\mathbf{p}$ is in $span\{\mathbf{b_1}, \mathbf{b_2}, \cdots, \mathbf{b_n}\}$ iff there **exists** c_i such that

$$\mathbf{p} = c_1\mathbf{b_1} + c_2\mathbf{b_2} + \ldots + c_n\mathbf{b_n}$$

Example: $span\{1, t, t^2\}$ = polynomials of degree at most 2.

A polynomial $p(t)$ is in the span of $\{1, t, t^2\}$ if and only if there **exists** a solution for a, b, c to the equation

$$p(t) = a1 + bt + ct^2$$

EXISTENCE

at least one sol'n

Example 1: $2 + t^3$ is not in the span of $\{1, t, t^2\}$ since there does not exist a, b, c such that

$$2 + t^3 = a + bt + ct^2$$

Example 2: $1 + 2t + 3t^2$ is in the span of $\{1, t, t^2\}$ since there exists a, b, c such that

$$1 + 2t + 3t^2 = a1 + bt + ct^2$$

In particular, $a = 1, b = 2, c = 3$ is a solution.

UNIQUENESS

at most one sol'n

$\mathbf{b_1}, \ldots, \mathbf{b_n}$ are linearly independent iff

$$c_1\mathbf{b_1} + c_2\mathbf{b_2} + \ldots + c_n\mathbf{b_n} = 0 \Longrightarrow c_1 = \ldots = c_n = 0$$

or equivalently,

$\mathbf{b_1}, \ldots, \mathbf{b_n}$ are linearly independent iff

$$c_1\mathbf{b_1} + c_2\mathbf{b_2} + \ldots + c_n\mathbf{b_n} = d_1\mathbf{b_1} + d_2\mathbf{b_2} + \ldots + d_n\mathbf{b_n}$$

$$\Longrightarrow c_1 = d_1, c_2 = d_2, \ldots = c_n = d_n.$$

In other words, if a solution exists for the following equation, then the solution is **unique**:

$$\mathbf{p} = c_1\mathbf{b_1} + c_2\mathbf{b_2} + \ldots + c_n\mathbf{b_n}$$

unique representation

$p \in \text{span}\{b_1, \ldots b_n\}$

UNIQUENESS

Example 1:
$\mathbf{b_1} = (1,0,0)$, $\mathbf{b_2} = (0,1,0)$, $\mathbf{b_3} = (0,0,1)$.

$$(1,2,3) \neq (1,2,4).$$

If $(a,b,c) = \underline{(1,2,3)}$, then $\underline{a = 1,\ b = 2,\ c = 3}$.

unique soln

Example 2: $\mathbf{b_1} = \underline{1}$, $\mathbf{b_2} = \underline{t}$, $\mathbf{b_3} = \underline{t^2}$.

$3t^2 + 2t + 1$

$$1 + 2t + 3t^2 \neq 1 + 2t + 4t^2.$$

If $a1 + bt + ct^2 = \underline{1 + 2t + 3t^2}$, then $\underline{a = 1}$, $\underline{b = 2}$, $\underline{c = 3}$.

BASIS

$\{\mathbf{b_1}, \mathbf{b_2}, \cdots, \mathbf{b_n}\}$ is a basis for the vector space V if

1.) $span\{\mathbf{b_1}, \mathbf{b_2}, \cdots, \mathbf{b_n}\} = V$ and $\vec{p} \in V$

2.) $\{\mathbf{b_1}, \mathbf{b_2}, \cdots, \mathbf{b_n}\}$ is a linearly independent set.

In other words if $\mathbf{p} \in V$, then there **exists** a solution for c_i for the following equation and that solution is **unique**:

$$\mathbf{p} = c_1\mathbf{b_1} + c_2\mathbf{b_2} + ... + c_n\mathbf{b_n}$$

Example 1: $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is a basis for $\mathbb{R}^3$.

Example 2: $\{1, t, t^2\} =$ is a basis for the set of polynomials of degree at most 2.

Application: Partial Fractions

First factor denominator over $\mathbb{R}$

$$\underbrace{\int \frac{x+1}{(x+2)(x-3)}\,dx}_{LHS} = \underbrace{\int \frac{A}{x+2}\,dx + \int \frac{B}{x-3}\,dx}_{RHS}$$

$$\frac{4}{(x^2+1)(x-3)} = \underbrace{\frac{Ax+B}{x^2+1}}_{\substack{\text{deg } 1 \\ \text{deg } 2}} + \underbrace{\frac{C}{x-3}}_{\substack{\text{deg } 0 \\ \text{deg } 1}} + \frac{\overbrace{Dx^2+Ex+F}^{\text{deg } 2}}{x^3+x^2+1}$$

(x^3+x+1)

~~$(x-i)(x+i)$~~

deg 3

$$\frac{x^4+1}{(x^2+1)(x-3)^3} = \frac{Ax+B}{x^2+1} + \frac{C}{x-3} + \frac{D}{(x-3)^2} + \frac{E}{(x-3)^3}$$

deg 0   deg 1

deg 1

$$\frac{4}{(x^2+1)^2(x-3)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} + \frac{E}{x-3}$$

deg 1   deg 2

$$x-1 = x\underbrace{+1-1}_{\text{"}+0\text{"}}-1$$

Both deg 1 poly

Don't forget to simplify first

$$\frac{x^2-1}{(x+1)^2} = \frac{(x-1)(x+1)}{(x+1)^2} = \frac{x-1}{x+1} = \frac{x+1-1-1}{x+1}$$

$$= \frac{x+1-2}{x+1} = \frac{x+1}{x+1} + \frac{-2}{x+1} = 1 + \frac{-2}{x+1}$$

For partial fractions, the power in numerator must be less than the power in denominator.

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If power in numerator $\geq$ power in denominator, do long division first (or add a "0" and simplify algebraically).

Solve for A, B, C

Application: Partial Fractions

$$\cancel{(x^2+1)(x-3)}\;\frac{4}{\cancel{(x^2+1)(x-3)}} = \left(\frac{Ax+B}{x^2+1} + \frac{C}{x-3}\right)(x^2+1)(x-3)$$

If you don't like denominators, get rid of them:

$$4 = (Ax+B)(x-3) + C(x^2+1)$$

$$4 = Ax^2 + Bx - 3Ax - 3B + Cx^2 + C$$

combine like terms

$$\boxed{4 = (A+C)x^2 + (B-3A)x - 3B + C}$$

linear algebra

I.e.,

$$\boxed{0x^2 + \boxed{0}x + 4 = (A+C)x^2 + \boxed{(B-3A)}x - 3B + C}$$

$\{x^2, x, 1\}$ is lin indep

$\{x^2, x, 1\}$ is li

$$0x^2 + \boxed{0}x + 4 = (A+C)x^2 + (B-3A)x - 3B + C$$

$$\boxed{\text{Thus } 0 = A + C, \quad \boxed{0 = B - 3A}, \quad 4 = -3B + C.}$$

$C = -A, \quad B = 3A, \qquad 4 = -3(3A) + -A \Rightarrow$
$4 = -10A.$

Hence $A = -\frac{2}{5}, \quad B = 3(-\frac{2}{5}) = -\frac{6}{5}, \quad C = \frac{2}{5}.$

$$\text{Thus, } \frac{4}{(x^2+1)(x-3)} = \frac{-\frac{2}{5}x - \frac{6}{5}}{x^2+1} + \frac{\frac{2}{5}}{x-3}$$

$$= \boxed{\frac{-2x-6}{5(x^2+1)} + \frac{2}{5(x-3)}}$$

Note there are many correct ways to solve for A, B, C. For example, one can plug in $x = 3$ to quickly find C and then solve for A and B.

$$4 = (Ax + B)(x - 3) + C(x^2 + 1)$$

One can also use $\boxed{\text{matrices to solve linear eqns.}}$

$$4 = (\sim)(3-3) + C(9+1)$$

$$4 = 0 + 10C \Rightarrow C = \frac{4}{10} = \frac{2}{5}$$

Let $x = 0$