HW 5: Look at comments before submitting HW 6 as HW 6 will be graded more rigourously.

In class quizzes are available again. While I recommend doing them on each class day, the unofficial due date is Sunday (note there is no late penalty to allow schedule flexibility).

Linear matrix equation: $Ax = b$ ← equivalent to linear system of linear eqns

MATH 2700

Linear differential equation:

$$y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = g(t)$$

equation ⇒ =

Linear combination of vectors: $c_1\mathbf{v_1} + ... + c_n\mathbf{v_n}$

Linear combination of functions: $c_1\phi_\mathbf{1} + ... + c_n\phi_\mathbf{n}$

Linear Functions

Vector space of fns

soln set for homog

A function f is linear if $f(a\mathbf{x} + b\mathbf{y}) = af(\mathbf{x}) + bf(\mathbf{y})$

Or equivalently f is linear if
1.) $f(a\mathbf{x}) = af(\mathbf{x})$ and
2.) $f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$

LHS of linear eqn is a linear function

FYI Theorem: If f is linear function, then $f(\mathbf{0}) = \mathbf{0}$

Proof: $f(\mathbf{0}) = f(0 \cdot \mathbf{0}) = 0 \cdot f(\mathbf{0}) = \mathbf{0}$

Example 1a.) $f : R \to R$, $f(x) = 2x$ ✓

Proof:
$f(ax + by) = 2(ax + by) = 2ax + 2by = af(x) + bf(y)$

Example 1b.) $f : R \to R$, $f(x) = 2x + 3$ is NOT linear.

eqn of line, NOT LINEAR FN

Proof: $f(2 \cdot 0) = f(0) = 3$, but $2f(0) = 2 \cdot 3 = 6$.
Hence $f(2 \cdot 0) \neq 2f(0)$

Alternate Proof: $f(0 + 1) = f(1) = 5$, but
$f(0) + f(1) = 3 + 5 = 8$. Hence $f(0+1) \neq f(0) + f(1)$

Note confusing notation: Most lines, $f(x) = mx + b$ are not linear functions.

Question: When is a line, $f(x) = mx + b$, a linear function?

$b = 0$

$f(x) = mx$ is a linear fn

$f(ax + by) = m(ax + by) = amx + bmy = af(x) + bf(y)$

Ex 1: $f(x) = [2]x$

Ex 2: $f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Example 2.) $f : R^2 \to R^2$,
$f((x_1, x_2)) = (2x_1, x_1 + x_2)$ } matrix eqn

Proof: Let $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$ $A(rx + sy)$

$a\mathbf{x} + b\mathbf{y} = a(x_1, x_2) + b(y_1, y_2) = (ax_1, ax_2) + (by_1, by_2) =$
$(ax_1 + by_1, ax_2 + by_2)$

$$\begin{aligned}
f(&ax_1 + by_1, ax_2 + by_2) \\
&= (2(ax_1 + by_1), ax_1 + by_1 + ax_2 + by_2) \\
&= (2ax_1 + 2by_1, ax_1 + ax_2 + by_1 + by_2) \\
&= (2ax_1, ax_1 + ax_2) + (2by_1, by_1 + by_2) \\
&= a(2x_1, x_1 + x_2) + b(2y_1, y_1 + y_2) \\
&= af((x_1, x_2)) + bf((y_1, y_2))
\end{aligned}$$

Example 3.) D : set of all differential functions $\to$ set of all functions, $D(f) = f'$

Proof:
$D(\underline{af + bg}) = (af + bg)' = af' + bg' = aD(f) + bD(g)$

Example 4.) Given a, b real numbers,
I : set of all integrable functions on [a, b] $\to R$, $I(f) = \int_a^b f$

Proof: $I(sf + tg) = \int_a^b sf + tg = s\int_a^b f + t\int_a^b g = sI(f) + tI(g)$

Example 5.) The inverse of a linear function is linear (when the inverse exists). Math 2560 → ch 6

Suppose $f^{-1}(x) = c$, $f^{-1}(y) = d$. LaPLace transform

Then $f(c) = x$ and $f(d) = y$ and
$f(ac + bd) = af(c) + bf(d) = ax + by$.

Hence $f^{-1}(ax + by) = ac + bd = af^{-1}(x) + bf^{-1}(y)$.

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ch3

Example 6.) D : set of all twice differential functions $\to$ set of all functions, $L(f) = af'' + bf' + cf$

LHS of Linear DE

Proof:

$$L(sf + tg) = a(sf + tg)'' + b(sf + tg)' + c(sf + tg)$$

$$= saf'' + tag'' + sbf' + tbg' + scf + tcg$$

$$= s(af'' + bf' + cf) + t(ag'' + bg' + cg)$$

$$= sL(f) + tL(g)$$ ✓

ch3

$L(f) = \overbrace{af'' + bf' + cf}^{LHS} = 0$

Consequence 1: If ϕ_1, ϕ_2 are solutions to $af'' + bf' + cf = 0$, then $3\phi_1 + 5\phi_2$ is also a solution to $af'' + bf' + cf = 0$,

Pf: Since ϕ_1 & ϕ_2 are solns to $af'' + bf' + cf = 0$

$\Rightarrow a\phi_i'' + b\phi_i' + c\phi_i = 0$

Let $L(f) = af'' + bf' + cf$

$L(\phi_i) = a\phi_i'' + b\phi_i' + c\phi_i = 0$

$L(3\phi_1 + 5\phi_2) = 3L(\phi_1) + 5L(\phi_2) = 3\cdot 0 + 5\cdot 0 = 0$

Thm 3.2.2: If ϕ_1 and ϕ_2 are two solutions to a homogeneous linear differential equation

$$y'' + p(t)y' + q(t)y = 0$$

then $c_1\phi_1 + c_2\phi_2$ is also a solution to this linear differential equation.

Let $L(y) = y'' + p(t)y' + q(t)y$

Note L is a linear function

(special case of example 6

Note f is a soln $\Longleftrightarrow L(f) = 0$

$L(c_1\phi_1 + c_2\phi_2) = c_1 L(\phi_1) + c_2 L(\phi_2) = c_1(0) + c_2(0) = 0$

$\Rightarrow c_1\phi_1 + c_2\phi_2$ is a soln

Consequence 1: If ϕ_1, ϕ_2 are solutions to $af'' + bf' + cf = 0$, then $3\phi_1 + 5\phi_2$ is also a solution to $af'' + bf' + cf = 0$,

let $L(f) = af'' + bf' + cf$

Proof: Since ϕ_1, ϕ_2 are solutions to $af'' + bf' + cf = 0$, $L(\phi_1) = 0$ and $L(\phi_2) = 0$.

Hence $L(3\phi_1 + 5\phi_2) = 3L(\phi_1) + 5L(\phi_2)$

$$= 3(0) + 5(0) = 0.$$

Thus $3\phi_1 + 5\phi_2$ is also a solution to $af'' + bf' + cf = 0$

Thm 3.2.2: If ϕ_1 and ϕ_2 are two solutions to a homogeneous linear DE

LHS RHS

$$y'' + p(t)y' + q(t)y = 0 \quad (*)$$

$c_1\phi_1 + c_2\phi_2$ is also a solution to this linear DE.

Proof: $L(y) = y'' + p(t)y' + q(t)y$ is a linear function.

The function $y = h(t)$ is a solution to (*) iff $L(h) = 0$.

Since ϕ_i are solutions to (*), $L(\phi_i) = 0$ for $i = 1, 2$.

$$L(c_1\phi_1 + c_2\phi_2) = c_1L(\phi_1) + c_2L(\phi_2) = c_1(0) + c_2(0) = 0$$

RHS

Thus $y = c_1\phi_1 + c_2\phi_2$ is also a solution to (*).

3.2 Ⓐ Linear combinations of homog solns to homog solns are homog Linear homog DE

Ⓑ Wronskian

B1) The coef matrix used to find c_1 & c_2 when solving IVP

B2) $W(\phi_1, \phi_2)(t_0) \neq 0$ $\Longleftrightarrow$ IVP soln exists & is unique

Consequence 2: (relate to section 3.5

If ψ_1 is a solution to $af'' + bf' + cf = h$
and ψ_2 is a solution to $af'' + bf' + cf = k$,
then $3\psi_1+5\psi_2$ is a solution to $af''+bf'+cf = 3h+5k$,

NOT homog if $h \neq 0$, $k \neq 0$

$L(f) = af'' + bf' + cf$

Since ψ_1 is a solution to $\underbrace{af''+bf'+cf}_{LHS} = \underbrace{h}_{RHS}$, $L(\psi_1) = h$.

Since ψ_2 is a solution to $af''+bf'+cf = k$, $L(\psi_2) = k$.

Hence $\underbrace{L(3\psi_1 + 5\psi_2)}_{LHS} = 3L(\psi_1) + 5L(\psi_2)$

$$= 3h + 5k.$$

RHS

Thus $3\psi_1 + 5\psi_2$ is also a solution to

$$af'' + bf' + cf = 3h + 5k$$

Section 3.5: Solving linear non-homogeneous DE.

Example: Solve $y'' - 4y' - 5y = 4sin(3t)$

$\boxed{\text{Step 1}}$: Solve linear homog DE

$$y'' - 4y' - 5y = 0$$

$$r^2 - 4r - 5 = 0$$

$$(r-5)(r+1) = 0 \Rightarrow r = 5, -1$$

$\Rightarrow$ general homog solns

$$y = c_1 e^{5t} + c_2 e^{-t}$$

$\boxed{\text{Step 2:}}$ Example (cont): Solve $y'' - 4y' - 5y = 4sin(3t)$ Find 1 non homog soln to non homog linear DE

$$y'' - 4y' - 5y = 4\sin(3t)$$

Educated Guess (3.5 method) & then plug in

$$y = A\sin(3t) + B\cos(3t)$$

$$\rightarrow y' = 3A\cos(3t) - 3B\sin(3t)$$

$$y'' = -9A\sin(3t) - 9B\cos(3t)$$

$$(-9A\sin(3t) - 9B\cos(3t))$$

$$-4(-3B\sin(3t) + 3A\cos(3t))$$

$$-5(A\sin(3t) + B\cos(3t)) = 4\sin(3t)$$

Example (cont): Solve $\overbrace{y'' - 4y' - 5y}^{LHS} = \overbrace{4sin(3t)}^{RHS}$

$$(-9A - 4(-3B) - 5A)\sin(3t)$$
$$+ (-9B - 4(+3A) - 5B)\cos(3t)$$
$$= (-14A + 12B)\sin(3t)$$
$$+ (-14B - 12A)\cos(3t)$$
$$= 4\sin(3t) + 0\cos(3t)$$

$$\left.\begin{aligned} -14A + 12B &= 4 \\ -14B - 12A &= 0 \end{aligned}\right\} \text{etc}$$

Thm: Suppose $c_1\phi_1(t) + c_2\phi_2(t)$ is a general solution to

$$ay'' + by' + cy = 0,$$

If ψ is a solution to

$$ay'' + by' + cy = g(t) \ [*],$$

Then $\psi + c_1\phi_1(t) + c_2\phi_2(t)$ is also a solution to [*].

Moreover if γ is also a solution to [*], then there exist constants c_1, c_2 such that

$$\gamma = \psi + c_1\phi_1(t) + c_2\phi_2(t)$$

Or in other words, $\psi + c_1\phi_1(t) + c_2\phi_2(t)$ is a general solution to [*].

Proof:

Define $L(f) = af'' + bf' + cf$.

Recall L is a linear function.

Let $h = c_1\phi_1(t) + c_2\phi_2(t)$. Since h is a solution to the differential equation, $ay'' + by' + cy = 0$, Since ψ is a solution to $ay'' + by' + cy = g(t)$,

We will now show that $\psi + c_1\phi_1(t) + c_2\phi_2(t) = \psi + h$ is also a solution to [*].

Since γ a solution to $ay'' + by' + cy = g(t)$,

We will first show that $\gamma - \psi$ is a solution to the differential equation $ay'' + by' + cy = 0$.

Since $\gamma - \psi$ is a solution to $ay'' + by' + cy = 0$ and

$c_1\phi_1(t) + c_2\phi_2(t)$ is a general solution to
$$ay'' + by' + cy = 0,$$

there exist constants c_1, c_2 such that

$\gamma - \psi =$ ______________________________

Thus $\gamma = \psi + c_1\phi_1(t) + c_2\phi_2(t)$.