HW 5: Look at comments before submitting HW 6 as HW 6 will be graded more rigourously.

In class quizzes are available again. While I recommend doing them on each class day, the unofficial due date is Sunday (note there is no late penalty to allow schedule flexibility).

Linear matrix equation: $Ax = b$

Linear differential equation:

$$y^{(n)} + p_1(t)y^{(n-1)} + ... + p_{n-1}(t)y' + p_n(t)y = g(t)$$

Linear combination of vectors: $c_1\mathbf{v_1} + ... + c_n\mathbf{v_n}$

Linear combination of functions: $c_1\phi_{\mathbf{1}} + ... + c_n\phi_{\mathbf{n}}$

Linear Functions

A function f is linear if $f(a\mathbf{x} + b\mathbf{y}) = af(\mathbf{x}) + bf(\mathbf{y})$

Or equivalently f is linear if
1.) $f(a\mathbf{x}) = af(\mathbf{x})$ and
2.) $f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$

Theorem: If f is linear, then $f(\mathbf{0}) = \mathbf{0}$

Proof: $f(\mathbf{0}) = f(0 \cdot \mathbf{0}) = 0 \cdot f(\mathbf{0}) = \mathbf{0}$

Example 1a.) $f : R \to R$, $f(x) = 2x$

Proof:
$f(ax + by) = 2(ax + by) = 2ax + 2by = af(x) + bf(y)$

Example 1b.) $f : R \to R$, $f(x) = 2x + 3$ is NOT linear.

Proof: $f(2 \cdot 0) = f(0) = 3$, but $2f(0) = 2 \cdot 3 = 6$.
Hence $f(2 \cdot 0) \neq 2f(0)$

Alternate Proof: $f(0 + 1) = f(1) = 5$, but
$f(0) + f(1) = 3 + 5 = 8$. Hence $f(0 + 1) \neq f(0) + f(1)$

Note confusing notation: Most lines, $f(x) = mx + b$ are not linear functions.

Question: When is a line, $f(x) = mx + b$, a linear function?

Example 2.) $f : R^2 \to R^2$,

$$f((x_1, x_2)) = (2x_1, x_1 + x_2)$$

Proof: Let $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$

$a\mathbf{x}+b\mathbf{y} = a(x_1, x_2)+b(y_1, y_2) = (ax_1, ax_2)+(by_1, by_2) = (ax_1 + by_1, ax_2 + by_2)$

$f(ax_1 + by_1, ax_2 + by_2)$

$$= (2(ax_1 + by_1), ax_1 + by_1 + ax_2 + by_2)$$

$$= (2ax_1 + 2by_1, ax_1 + ax_2 + by_1 + by_2)$$

$$= (2ax_1, ax_1 + ax_2) + (2by_1, by_1 + by_2)$$

$$= a(2x_1, x_1 + x_2) + b(2y_1, y_1 + y_2)$$

$$= af((x_1, x_2)) + bf((y_1, y_2))$$

Example 3.) D : set of all differential functions $\to$ set of all functions, $D(f) = f'$

Proof:
$D(af + bg) = (af + bg)' = af' + bg' = aD(f) + bD(g)$

Example 4.) Given a, b real numbers,
I : set of all integrable functions on [a, b] $\to$ R , $I(f) = \int_a^b f$

Proof: $I(sf + tg) = \int_a^b sf + tg = s\int_a^b f + t\int_a^b g = sI(f) + tI(g)$

Example 5.) The inverse of a linear function is linear (when the inverse exists).

Suppose $f^{-1}(x) = c$, $f^{-1}(y) = d$.

Then $f(c) = x$ and $f(d) = y$ and
$f(ac + bd) = af(c) + bf(d) = ax + by$.

Hence $f^{-1}(ax + by) = ac + bd = af^{-1}(x) + bf^{-1}(y)$.

Example 6.) D : set of all twice differential functions $\to$ set of all functions, $L(f) = af'' + bf' + cf$

Proof:

$$\begin{aligned}
L(sf + tg) &= a(sf + tg)'' + b(sf + tg)' + c(sf + tg) \\
&= saf'' + tag'' + sbf' + tbg' + scf + tcg \\
&= s(af'' + bf' + cf) + t(ag'' + bg' + cg) \\
&= sL(f) + tL(g)
\end{aligned}$$

Consequence 1: If ϕ_1, ϕ_2 are solutions to $af'' + bf' + cf = 0$, then $3\phi_1 + 5\phi_2$ is also a solution to $af'' + bf' + cf = 0$,

---

Thm 3.2.2: If ϕ_1 and ϕ_2 are two solutions to a <u>homogeneous</u> linear differential equation

$$y'' + p(t)y' + q(t)y = 0$$

then $c_1\phi_1 + c_2\phi_2$ is also a solution to this linear differential equation.

Consequence 1: If ϕ_1, ϕ_2 are solutions to $af'' + bf' + cf = 0$, then $3\phi_1 + 5\phi_2$ is also a solution to $af'' + bf' + cf = 0$,

Proof: Since ϕ_1, ϕ_2 are solutions to $af''+bf'+cf = 0$, $L(\phi_1) = 0$ and $L(\phi_2) = 0$.

Hence $L(3\phi_1 + 5\phi_2) = 3L(\phi_1) + 5L(\phi_2)$

$$= 3(0) + 5(0) = 0.$$

Thus $3\phi_1+5\phi_2$ is also a solution to $af''+bf'+cf = 0$

Thm 3.2.2: If ϕ_1 and ϕ_2 are two solutions to a <u>homogeneous</u> linear DE

$$y'' + p(t)y' + q(t)y = 0 \quad (*)$$

$c_1\phi_1 + c_2\phi_2$ is also a solution to this linear DE.

Proof: $L(y) = y'' + p(t)y' + q(t)y$ is a linear function.

The function $y = h(t)$ is a solution to (*) iff $L(h) = 0$.

Since ϕ_i are solutions to (*), $L(\phi_i) = 0$ for $i = 1, 2$.

$$L(c_1\phi_1 + c_2\phi_2) = c_1L(\phi_1) + c_2L(\phi_2) = c_1(0) + c_2(0) = 0$$

Thus $y = c_1\phi_1 + c_2\phi_2$ is also a solution to (*).

Consequence 2:
If ψ_1 is a solution to $af'' + bf' + cf = h$
and ψ_2 is a solution to $af'' + bf' + cf = k$,
then $3\psi_1 + 5\psi_2$ is a solution to $af'' + bf' + cf = 3h + 5k$,

Since ψ_1 is a solution to $af'' + bf' + cf = h$, $L(\psi_1) = h$.

Since ψ_2 is a solution to $af'' + bf' + cf = k$, $L(\psi_2) = k$.

Hence $L(3\psi_1 + 5\psi_2) = 3L(\psi_1) + 5L(\psi_2)$

$$= 3h + 5k.$$

Thus $3\psi_1 + 5\psi_2$ is also a solution to

$$af'' + bf' + cf = 3h + 5k$$

Section 3.5: Solving linear non-homogeneous DE.

Example: Solve $y'' - 4y' - 5y = 4sin(3t)$

Example (cont): Solve $y'' - 4y' - 5y = 4sin(3t)$

Example (cont): Solve $y'' - 4y' - 5y = 4sin(3t)$

Thm: Suppose $c_1\phi_1(t) + c_2\phi_2(t)$ is a general solution to

$$ay'' + by' + cy = 0,$$

If ψ is a solution to

$$ay'' + by' + cy = g(t) \ [*],$$

Then $\psi + c_1\phi_1(t) + c_2\phi_2(t)$ is also a solution to [*].

Moreover if γ is also a solution to [*], then there exist constants c_1, c_2 such that

$$\gamma = \psi + c_1\phi_1(t) + c_2\phi_2(t)$$

Or in other words, $\psi + c_1\phi_1(t) + c_2\phi_2(t)$ is a general solution to [*].

Proof:

Define $L(f) = af'' + bf' + cf$.

Recall L is a linear function.

Let $h = c_1\phi_1(t) + c_2\phi_2(t)$. Since h is a solution to the differential equation, $ay'' + by' + cy = 0$, Since ψ is a solution to $ay'' + by' + cy = g(t)$,

We will now show that $\psi + c_1\phi_1(t) + c_2\phi_2(t) = \psi + h$ is also a solution to [*].

Since γ a solution to $ay'' + by' + cy = g(t)$,

We will first show that $\gamma - \psi$ is a solution to the differential equation $ay'' + by' + cy = 0$.

Since $\gamma - \psi$ is a solution to $ay'' + by' + cy = 0$ and

$c_1\phi_1(t) + c_2\phi_2(t)$ is a general solution to

$$ay'' + by' + cy = 0,$$

there exist constants c_1, c_2 such that

$$\gamma - \psi = \rule{8cm}{0.4pt}$$

Thus $\gamma = \psi + c_1\phi_1(t) + c_2\phi_2(t)$.