Solving polynomial equations:

$$\text{Example: } 1r^3 + r^2 + 3r + 10 = 0$$

$y''' + y'' + 3y' + 10y = 0$

Plug in $r = \pm 1, \pm 2, \pm 5, \pm 10$ to see if any of these are solns:

$$(\pm 1)^3 + (\pm 1)^2 + 3(\pm 1) + 10 \neq 0$$

$$(\pm 2)^3 + (\pm 2)^2 + 3(\pm 2) + 10 \ ? =? 0$$

$$(-2)^3 + (-2)^2 + 3(-2) + 10 = -8 + 4 - 6 + 10 = 0$$

Thus $(r - (-2))$ is a factor of $r^3 + r^2 + \boxed{3r} + 10$

Hence $r^3 + r^2 + 3r + 10 = (r + 2)(r^2 + \underline{-1}\ r + \boxed{5})$

$10/2$

To find the coefficient of r in the above, you can do so by (1) long division, (2) inspection, (3) using variable x

$$r^3 + r^2 + 3r + 10 = (r+2)(r^2 + \underline{\boxed{x}}\, r + 5)$$

$$(r+2)(r^2 + \underline{\boxed{x}}\, r + 5) = r^3 + (2+x)r^2 + (2x+5)r + 10$$

$$r^3 \quad + \quad \underline{1}\, r^2 \quad + \quad \underline{3r} \quad + \quad 10$$

Thus $2 + x = 1$ and hence $x = -1$

Check: $2x + 5 = 2(-1) + 5 = 3$

Hence $r^3 + r^2 + 3r + 10 = (r+2)(r^2 - r + 5) = 0$

quadratic formula

Thus $r = -2, \frac{1 \pm \sqrt{1-20}}{2}$.

Thus $r = -2, \frac{1 \pm i\sqrt{19}}{2}$.

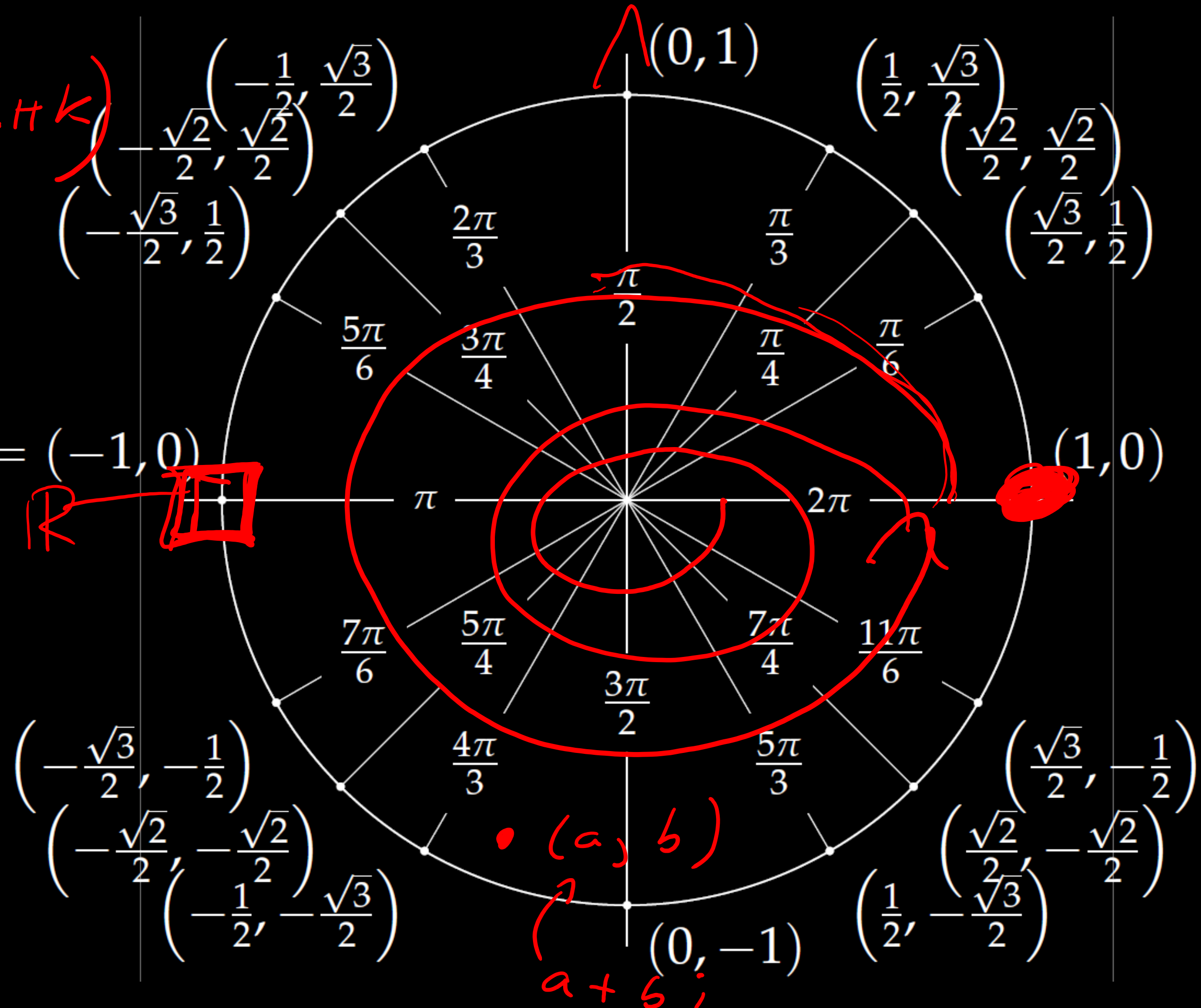
In special cases, you can use the unit circle.

zero

$r^n = 1 = e^{i0} = e^{i(0+2\pi k)}$

$r^n = -1 = e^{i\pi} = e^{i(\pi + 2\pi k)}$

nth root of unity



$(\cos\theta, \sin\theta)$

$\sin(0) = \frac{\sqrt{0}}{2} = 0$

$\sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{1}}{2} = \frac{1}{2}$

$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$\sin\left(\frac{\pi}{2}\right) = \frac{\sqrt{4}}{2} = 1$

$\cos(0) = 1$

$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

$\cos\left(\frac{\pi}{2}\right) = 0$

In special cases, you can use the unit circle.

4rt degree poly $\Rightarrow$ 4 roots

$y'''' + y = 0$

Ex: $\boxed{r^4 + 1} = 0$ implies $\Rightarrow r^4 = -1$

$$r = (-1)^{\frac{1}{4}} = (-1 + 0i)^{\frac{1}{4}} = (e^{i\pi})^{\frac{1}{4}} = (e^{i(\pi + 2\pi k)})^{\frac{1}{4}}$$

$$k = 0: \quad e^{\frac{i\pi}{4}} = cos(\tfrac{\pi}{4}) + isin(\tfrac{\pi}{4}) = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

$$k = 1: \quad e^{\frac{3i\pi}{4}} = cos(\tfrac{3\pi}{4}) + isin(\tfrac{3\pi}{4}) = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

$$k = 2: \quad e^{\frac{5i\pi}{4}} = cos(\tfrac{5\pi}{4}) + isin(\tfrac{5\pi}{4}) = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$

$$k = 3: \quad e^{\frac{7i\pi}{4}} = cos(\tfrac{7\pi}{4}) + isin(\tfrac{7\pi}{4}) = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$

$$r = (-1)^{\frac{1}{4}} = (-1+0i)^{\frac{1}{4}} = (e^{i\pi})^{\frac{1}{4}} = (e^{i(\pi+2\pi k)})^{\frac{1}{4}}$$

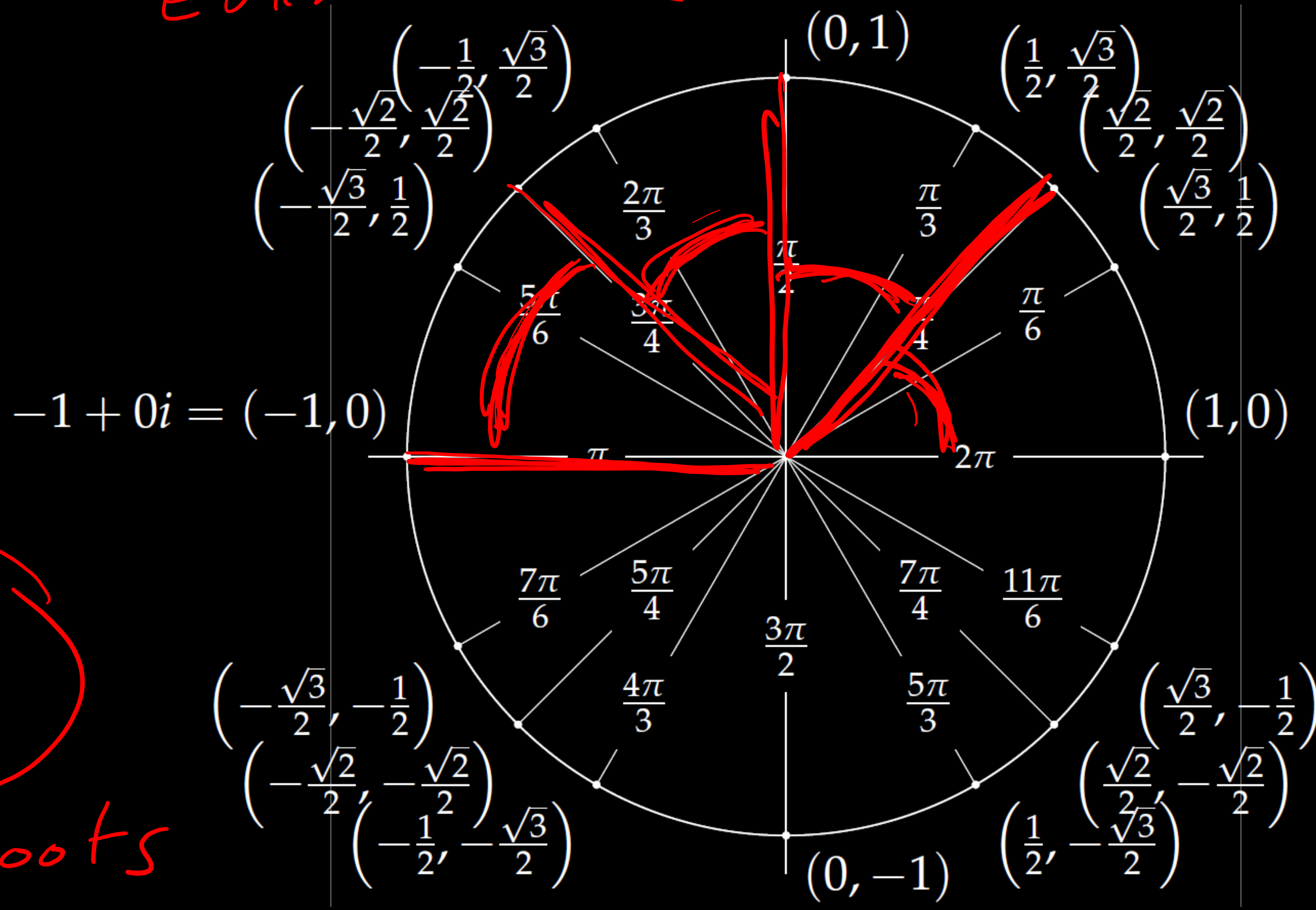
$$k = 0: \quad r_1 = e^{\frac{i\pi}{4}} = \underbrace{cos(\tfrac{\pi}{4}) + isin(\tfrac{\pi}{4})}_{\text{Euler's formula}} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} = r_1$$

$$e^{\frac{i(\pi+2\pi k)}{4}}$$

$k = 0$

$$e^{\frac{i\pi}{4}} = r_1$$

one of our 4 roots



$$r = (-1)^{\frac{1}{4}} = (-1+0i)^{\frac{1}{4}} = (e^{i\pi})^{\frac{1}{4}} = (e^{i(\pi+2\pi k)})^{\frac{1}{4}}$$

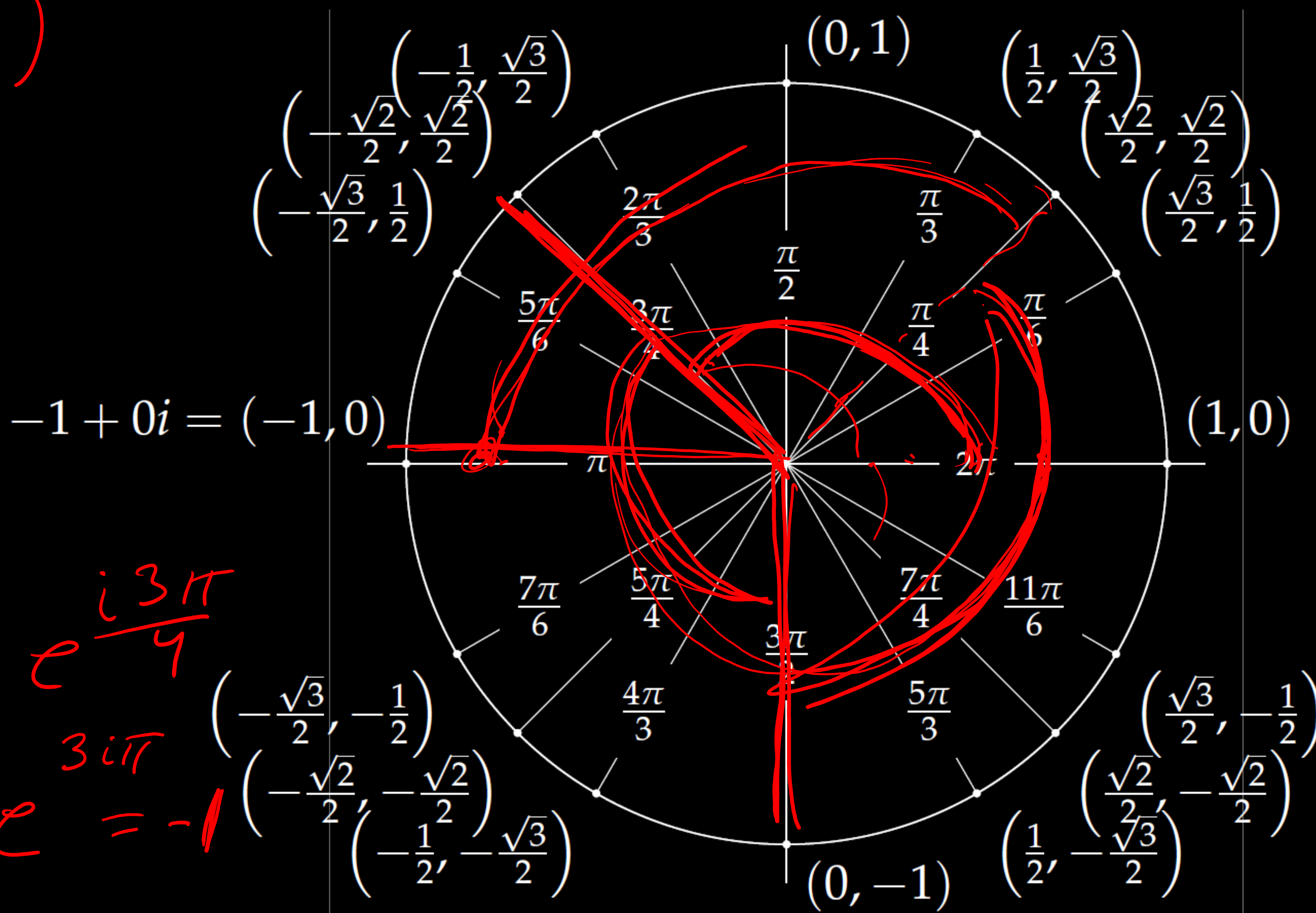
$$k = 1: \qquad e^{\frac{3i\pi}{4}} = cos(\frac{3\pi}{4}) + isin(\frac{3\pi}{4}) = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} = r_2$$

$$e^{\frac{i(\pi+2\pi K)}{4}}$$

$K = 1$

$$e^{\frac{i(\pi+2\pi)}{4}} = e^{\frac{i3\pi}{4}}$$

$$\left(e^{\frac{i3\pi}{4}}\right)^4 = e^{3i\pi} = -1$$

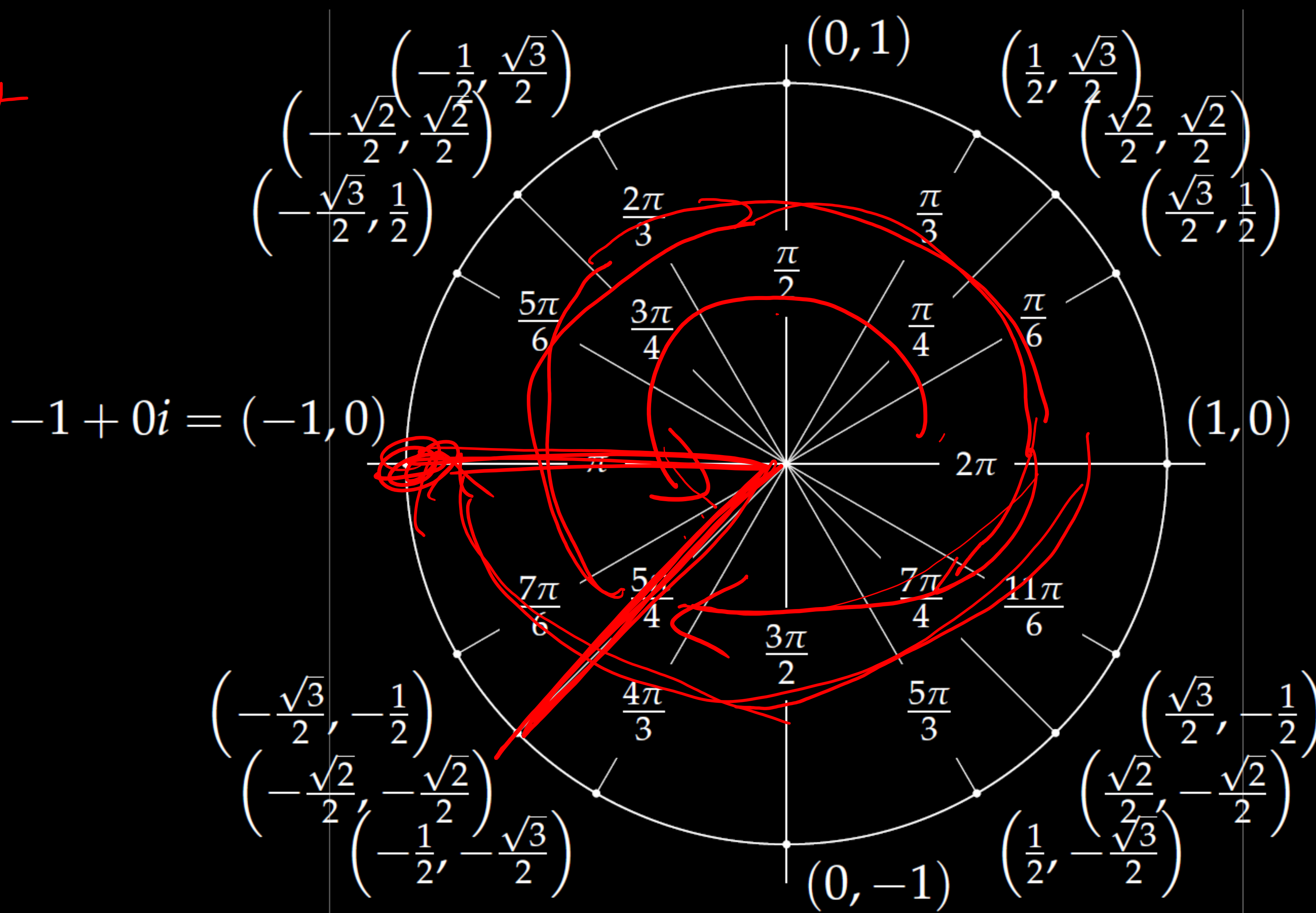


$$r = (-1)^{\frac{1}{4}} = (-1+0i)^{\frac{1}{4}} = (e^{i\pi})^{\frac{1}{4}} = (e^{i(\pi+2\pi k)})^{\frac{1}{4}}$$

$$k = 2: \qquad e^{\frac{5i\pi}{4}} = cos(\frac{5\pi}{4}) + isin(\frac{5\pi}{4}) = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} = r_3$$

$$4$$

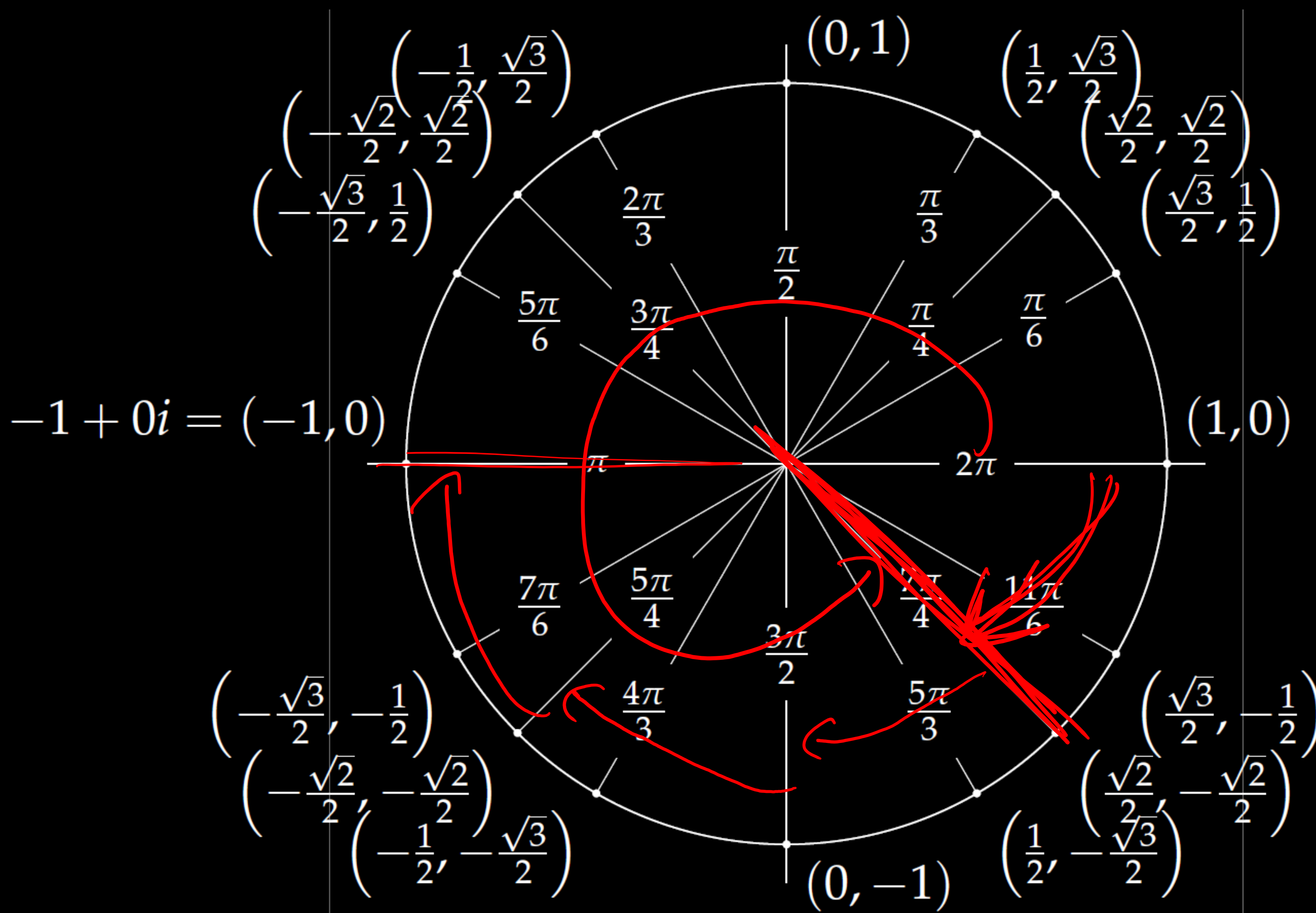
$$e^{\frac{-3i\pi}{4}}$$



$$r = (-1)^{\frac{1}{4}} = (-1 + 0i)^{\frac{1}{4}} = (e^{i\pi})^{\frac{1}{4}} = (e^{i(\pi + 2\pi k)})^{\frac{1}{4}}$$

$$\underline{k = 3:} \qquad e^{\frac{7i\pi}{4}} = cos(\tfrac{7\pi}{4}) + isin(\tfrac{7\pi}{4}) = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} = r_4$$

$$= e^{\frac{-i\pi}{4}}$$



$$r = (-1)^{\frac{1}{4}} = (-1+0i)^{\frac{1}{4}} = \left(e^{i\pi}\right)^{\frac{1}{4}} = \left(e^{i(\pi+2\pi k)}\right)^{\frac{1}{4}}$$

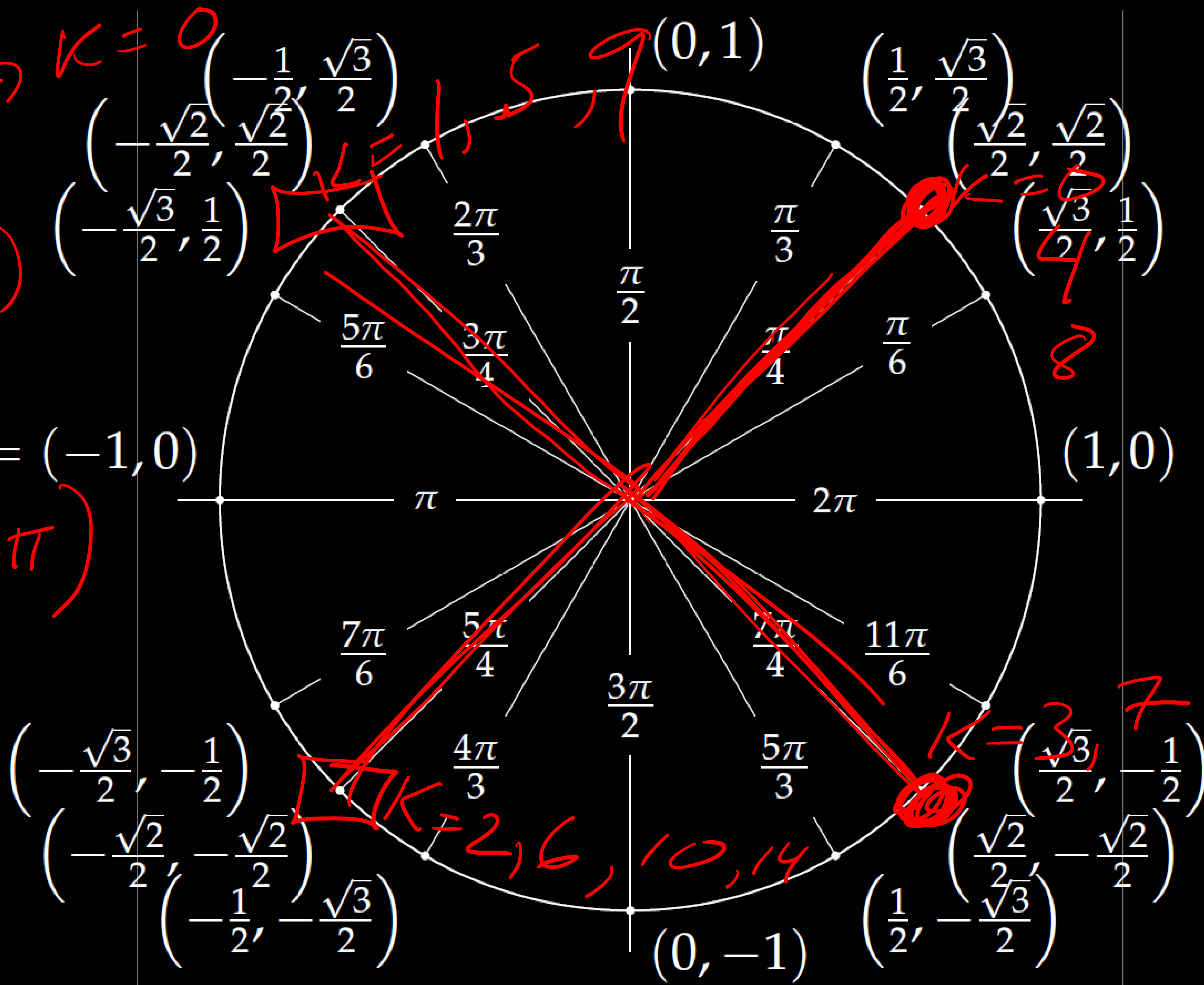
$K = 4$

$$e^{\frac{i(\pi+8\pi k)}{4}}$$

$-1 + 0i = (-1, 0)$

$$= e^{i\left[\left(\frac{\pi}{4}\right)+2\pi\right]}$$

$$= e^{i\frac{\pi}{4}}$$



In special cases, you can use the unit circle.

Ex: $r^4 + 1 = 0$ implies ← looking for 4 roots

$$r = (-1)^{\frac{1}{4}} = (-1 + 0i)^{\frac{1}{4}} = (e^{i\pi})^{\frac{1}{4}} = (e^{i(\pi + 2\pi k)})^{\frac{1}{4}}$$

$$k = 0: \quad e^{\frac{i\pi}{4}} = cos(\tfrac{\pi}{4}) + isin(\tfrac{\pi}{4}) = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} = r_1$$

$$k = 1: \quad e^{\frac{3i\pi}{4}} = cos(\tfrac{3\pi}{4}) + isin(\tfrac{3\pi}{4}) = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} = r_2$$

$$k = 2: \quad e^{\frac{5i\pi}{4}} = cos(\tfrac{5\pi}{4}) + isin(\tfrac{5\pi}{4}) = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} = r_3$$

$$k = 3: \quad e^{\frac{7i\pi}{4}} = cos(\tfrac{7\pi}{4}) + isin(\tfrac{7\pi}{4}) = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} = r_4$$

Example: Solve $\boxed{y^{(iv)} + y = 0}$

$r^{2n} = -1$

$y = e^{rt}$ implies $\boxed{r^4 + 1 = 0}$ and thus

$$r = \frac{\sqrt{2}}{2} \pm i\frac{\sqrt{2}}{2} \text{ and } r = -\frac{\sqrt{2}}{2} \pm i\frac{\sqrt{2}}{2}$$

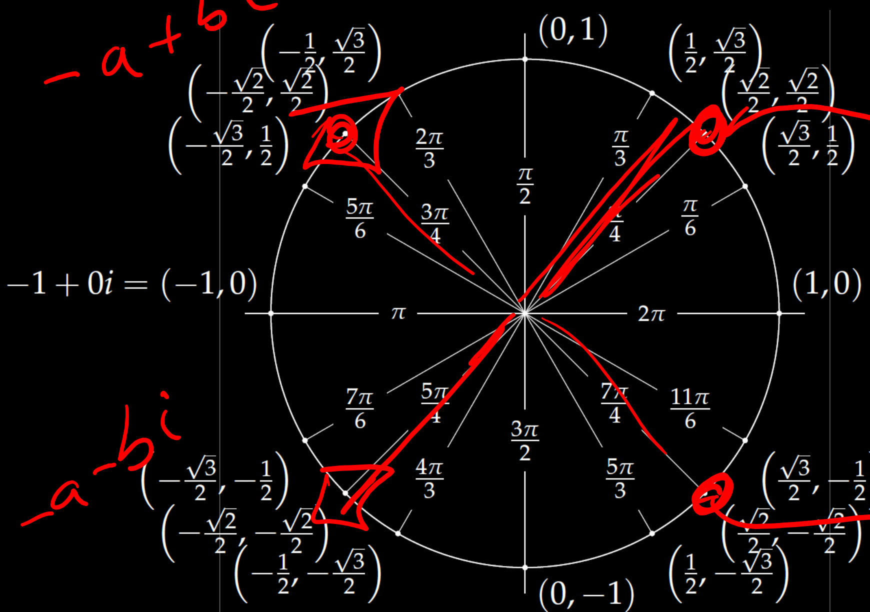
complex conjug

Thus general homogeneous solution is

$$y = c_1 e^{\frac{\sqrt{2}}{2}t} cos(\frac{\sqrt{2}}{2}t) + c_2 e^{\frac{\sqrt{2}}{2}t} sin(\frac{\sqrt{2}}{2}t)$$

$$+ c_3 e^{\boxed{-\frac{\sqrt{2}}{2}t}} cos(\frac{\sqrt{2}}{2}t) + c_4 e^{\boxed{-\frac{\sqrt{2}}{2}t}} sin(\frac{\sqrt{2}}{2}t)$$

-a+bi



-a-bi

a+bi

a-bi

complex conjugates

Example: Solve $y''' - 8y = 0$, $y(0) = 0$, $y'(0) = 0$, $y''(0) = 0$.

$r^3 - 8 = 0 \Rightarrow (r-2)(r^2 + \_\_ + 4)$

$\Rightarrow r^3 = 8 \Rightarrow r^{\cancel{3}} = \sqrt[3]{8} = 2 = 2(1)^{1/3}$

$\Rightarrow r = 2,\ a+bi,\ a-bi$

3 roots

need 3 roots

3 roots of unity for $(1)^{1/3}$

then multiply by $2 \Rightarrow r = 2(1)^{1/3}$

Example: Solve $y''' - 8y = 0$, $y(0) = 0$, $y'(0) = 0$, $y''(0) = 0$.

$r = 2(1)^{1/3} \leftarrow$ 2 times root of unity

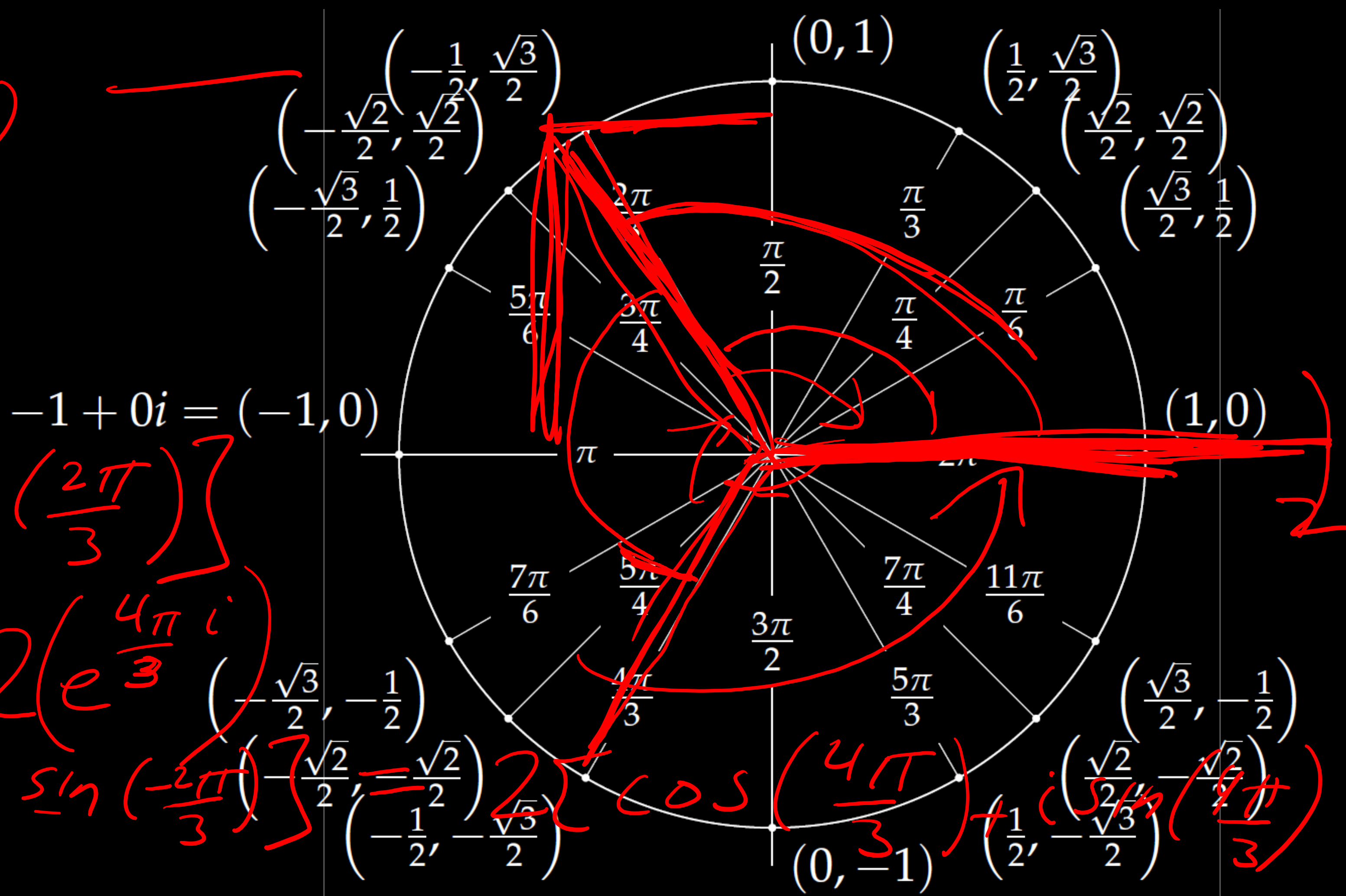
$r_1 = 2, \_\_\_\_, \_\_\_\_$

$r_2 = 2\left(e^{\frac{2\pi i}{3}}\right)$

$= 2\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right]$

$r_3 = 2\left(e^{-\frac{2\pi i}{3}}\right) = 2\left(e^{\frac{4\pi i}{3}}\right)$

$= 2\left[\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right] = 2\left[\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right]$



$$r = 2(1)^{1/3} = 2\left(e^{i0}\right)^{1/3}$$

$$= 2e^{\left(i\frac{(0+2\pi k)}{3}\right)}$$

$$k = 0 \Rightarrow r_1 = 2e^{0} = 2$$

$$k = 1 \Rightarrow r_2 = 2e^{\frac{2\pi i}{3}} = 2\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right]$$

$$k = 2 \Rightarrow r_2 = 2e^{\frac{4\pi i}{3}} = 2\left[\cos\frac{4\pi}{3} + i\sin\left(\frac{4\pi}{3}\right)\right]$$

or $k = -1$

$$r_1 = 2 \Rightarrow y = e^{2t}$$

$$r_2 = 2\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right]$$

$$= \cancel{2}\left(-\frac{1}{\cancel{2}} + i\left(\frac{\sqrt{3}}{\cancel{2}}\right)\right)$$

$$r_3 = \cancel{2}\left[-\frac{1}{\cancel{2}} - i\left(\frac{\sqrt{3}}{\cancel{2}}\right)\right]$$

$$\left.\begin{matrix} r_2 \\ r_3 \end{matrix}\right\} -1 \pm i\sqrt{3}$$

$(\cos\theta, \sin(\theta))$

$r_1 = 2 \Rightarrow e^{2t}$

$r_2, r_3 = \underbrace{\left(-1 \pm i\sqrt{3}\right)}_{} \Rightarrow \underbrace{e^{-t}\left(\cos(\sqrt{3}\,t) + \sin(\sqrt{3}\,t)\right)}_{}$

general soln is

$$y = c_1 e^{2t} + c_2 e^{-t} \cdot \cos(\sqrt{3}\,t) + c_3 e^{-t} \cdot \sin(\sqrt{3}\,t)$$

$y' =$

$y'' =$

IVP

$y(0) = 0$, $y'(0) = 0$ $y''(0) = 0$ t_0

plug in to find c_1, c_2 c_3

3rd order linear DE $\Rightarrow$ 3 unknowns

so we need 3 initial values

$\underbrace{y(t_0) = y_0}_{\text{point}}$, $\underbrace{y'(t_0) = y_1}_{\text{slope}}$, $\underbrace{y''(t_0) = y_2}_{\text{concavity}}$

left to viewer

$$y'' + p(t)\,y' + q(t)\,y = g(t)$$

$$\frac{a}{a}\,y'' + \boxed{\frac{b}{a}}\,y' + \boxed{\frac{c}{a}}\,y = \frac{g(t)}{a}$$

$t_0 \in (a,b)$

If p, q, g are cont on (a,b)

$\Rightarrow$ IVP $y(t_0) = y_0$ and $y'(t_0) = y_1$

has unique soln

g cont on (a,b) st $t_0 \in (a,b)$

g cont at t_0 is NOT suffici

F

Not IVP

$y(t_0) = y_0,\ y(t_1) = y_1$ F

$y(t_1) = y_0 \quad y'(t_1) = y_1$ F