Week 9		
10/19	Abel, 7.3, 7.1 (pdf), annotated (pdf)	HW 8 (due Sunday 10/25) 4.1: 1, 4, 5, 7, 16 4.2: 4, 13, 16 4.3: 2, 7, 10 - 13 7.3: 13, 14, 15 7.1 (use matrix form): 3, 4, 5, 6, 12
10/21	ch 3, 4, 7 defns, 7.1, 7.2	
10/23	$\boxed{\text{Review}}$ ← induction, etc	Quiz 3 (due Sunday 10/25) [3 points, unlimited attempts]
Week 10		
10/26	$\boxed{\text{Problem session}}$	Tentative HW 9 (due Friday 10/30) and ??
10/28	Exam 2	

Tentative: 60 pts MC/TF and induction
blah 2 (20 pt) questions

Abel's theorem: if ϕ_i are homogeneous solutions to an nth order linear DE,

$$1 \quad y^{(n)} + p_1(t)y^{(n-1)} + \ldots + p_{n-1}(t)y' + p_n(t)y = 0$$

then $W(\phi_1, \phi_2, \ldots, \phi_n)(t) = \underbrace{ce^{-\int p_1(t)dt}}$ for some constant c

Ex: Find the Wronskian of a fundamental set of solutions of the DE

$y'' + 5y' = 0$

Method 1: Find homogeneous solution

$r^2 + 5r = 0$ implies $r = 0, -5$

homog sol'n $y = c_1 e^{0t} + c_2 e^{-5t} = c_1(1) + c_2 e^{-5t} = c_1 + c_2 e^{-5t}$

A fundamental set of solutions: $\underline{\{1, e^{-5t}\}}$

Wronskian $= W(1, e^{-5t})(t) = det \begin{pmatrix} 1 & e^{-5t} \\ 0 & -5e^{-5t} \end{pmatrix} = -5e^{-5t}$

Method 2: Abel's theorem: Wronskian $= ce^{-\int p_1(t)dt}$

$1y'' + 5y' = 0$ implies $p_1(t) = 5$.

Thus Wronskian $= W(1, e^{-5t})(t) = ce^{-\int 5dt} = \underbrace{ce^{-5t}}_{\text{HW answer}}$

Method 2: Abel's theorem: Wronskian $= ce^{-\int p_1(t)dt}$

$y'' + 5y' = 0$ implies $p_1(t) = 5$.

Thus Wronskian $= W(1, e^{-5t})(t) = ce^{-\int 5dt} = ce^{-5t}$

A fundamental set of solutions: $\{1, e^{-5t}\}$

Wronskian $= W(1, e^{-5t})(t) = det \begin{pmatrix} 1 & e^{-5t} \\ 0 & -5e^{-5t} \end{pmatrix} = -5e^{-5t}$

$W(0) = \begin{pmatrix} 1 & 1 \\ 0 & -5 \end{pmatrix} = -5$

Method 2: Abel's theorem: Wronskian $= ce^{-\int p_1(t)dt}$

up to a multiple

$y'' + 5y' = 0$ implies $p_1(t) = 5$.

Thus Wronskian $= W(1, e^{-5t})(t) = ce^{-\int 5dt} = ce^{-5t}$

$W(0) = -5$

$\shortparallel$

$W(0) = ce^{0}$

$-5 = ce^{0} \Rightarrow c = -5$

If know $W(t_0)$, can solve for c

$W = -5e^{-5t}$

A fund soln set $= \{1, e^{-5t}\}$ $y = c_1(1) + c_2 e^{-5t}$

Any pair of l.i solns will be a fund soln set

$$y = c_1(2) + c_2 e^{-5t}$$

Another fund soln set is $\{2, e^{-5t}\}$

$$W(2, e^{-5t}) = \begin{vmatrix} 2 & e^{-5t} \\ 0 & -5e^{-5t} \end{vmatrix} = -10$$

$$y'' + y = 0 \Rightarrow r^2 + 1 = 0 \Rightarrow r = \pm i$$

fund soln set is $\{ e^{it}, e^{-it} \}$

A <u>better</u> fund soln set is $\{ \cos(t), \sin(t) \}$

Chapter 7: Systems of Linear DE

1) Review basis Linear Algebra

Defn: A set V together with two operations, called addition and scalar multiplication is a **vector space** if the following vector space axioms are satisfied for all vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ in V and all scalars, c, d in R.

vector $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

Vector space axioms:

You can do algebra

a.) $\mathbf{u} + \mathbf{v}$ is in V

b.) $c\mathbf{u}$ is in V

c.) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

d.) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

e.) There is a vector, denoted by $\mathbf{0}$, in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u}$ in V

f.) For each $\mathbf{u}$ in V, there is an element, denoted by $-\mathbf{u}$, in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

g.) $(cd)\mathbf{u} = c(d\mathbf{u})$

h.) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

i.) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

j.) $1\mathbf{u} = \mathbf{u}$

Linear Algebra Review: Eigenvalues and Eigenvectors

Defn: λ is an **eigenvalue** of the linear transformation $T : V \to V$ if there exists a nonzero vector $\mathbf{x}$ in V such that $T(\mathbf{x}) = \lambda\mathbf{x}$. The vector $\mathbf{x}$ is said to be an **eigenvector** corresponding to the eigenvalue λ.

$A\vec{v} = \lambda \vec{v}$

$\Rightarrow \lambda$ is an e. value

w e. vector $\vec{v}$

Example: Let $T(\mathbf{x}) = \begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix}\mathbf{x}$.

Note $\underbrace{\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix}}_{A}\begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix} = -1\begin{bmatrix} -1 \\ 5 \end{bmatrix}$

Thus -1 is an eigenvalue of A and $\begin{bmatrix} -1 \\ 5 \end{bmatrix}$ is a corresponding eigenvector of A.

Note $\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Thus 5 is an eigenvalue of A and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a corresponding eigenvector of A.

Note $\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \end{bmatrix} \neq k \begin{bmatrix} 2 \\ 8 \end{bmatrix}$ for any k.

Thus $\begin{bmatrix} 2 \\ 8 \end{bmatrix}$ is NOT an eigenvector of A.

MOTIVATION:

FYI (not 3600)

Note $\begin{bmatrix} 2 \\ 8 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Thus $A \begin{bmatrix} 2 \\ 8 \end{bmatrix} = A\left(\begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = A \begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3A \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$= -1 \begin{bmatrix} -1 \\ 5 \end{bmatrix} + 3 \cdot 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \end{bmatrix}$$

Finding eigenvalues:

Suppose $A\mathbf{x} = \lambda\mathbf{x}$ (Note A is a SQUARE matrix).

Then $A\mathbf{x} = \lambda I\mathbf{x}$ where I is the identity matrix.

Thus $\lambda I\mathbf{x} - A\mathbf{x} = (\lambda I - A)\mathbf{x} = \mathbf{0}$ solve linear homogeneous system

Thus if $A\mathbf{x} = \lambda\mathbf{x}$ for a nonzero $\mathbf{x}$, then $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has a nonzero solution.

Thus $det(\lambda I - A)\mathbf{x} = 0$. or $\det(A - \lambda I) = 0$

non unique soln

Note that the eigenvectors corresponding to λ are the nonzero solutions of $(\lambda I - A)\mathbf{x} = \mathbf{0}$.

λ is an e.value if $\exists$ ∞ # of soln

Thus to find the eigenvalues of A and their corresponding eigenvectors:

Step 1: Find eigenvalues: Solve the equation

$$det(\lambda I - A) = 0 \text{ for } \lambda. \quad \text{or} \quad |A - \lambda I| = 0$$

Step 2: For each eigenvalue λ_0, find its corresponding eigenvectors by solving the homogeneous system of equations

$$(\lambda_0 I - A)\mathbf{x} = 0 \text{ for } \mathbf{x}. \qquad [A - \lambda I \mid 0]$$

Defn: $det(\lambda I - A) = 0$ is the **characteristic equation** of A.

$\hookrightarrow det(A - \lambda I) = 0$

Thm 3: The eigenvalues of an upper triangular or lower triangular matrix (including diagonal matrices) are identical to its diagonal entries.

Defn: The **eigenspace** corresponding to an eigenvalue λ_0 of a matrix A is the set of all solutions of $(\lambda_0 I - A)\mathbf{x} = \mathbf{0}$.

nullspace

Note: An eigenspace is a vector space

The vector $\mathbf{0}$ is always in the eigenspace.

The vector $\mathbf{0}$ is never an eigenvector.

$\det(A - 0I) = 0$

$\det A = 0$

The number 0 can be an eigenvalue.

A is NOT invertible

Thm: A square matrix is invertible if and only if $\lambda = 0$ is not an eigenvalue of A.

Example: Let $T(\mathbf{x}) = \begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \mathbf{x}$. ← not upper triangular

Step 1: $|A - \lambda I| = \begin{vmatrix} 4-\lambda & 1 \\ 5 & 0-\lambda \end{vmatrix}$

$= (4-\lambda)(-\lambda) - 5 = \lambda^2 - 4\lambda - 5$

$= (\lambda - 5)(\lambda + 1) = 0$

$\Rightarrow \underline{\lambda = 5}, \underline{\lambda = -1}$ ← $\boxed{\text{e. values}}$

not related to $\begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix}$

Example: Let $T(\mathbf{x}) = \begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \mathbf{x}$.

Step 2: Find an e. vector for each e. value

since our e. space is 1-dimensional

$\lambda = -1$: $A - \lambda I = \begin{bmatrix} 4-(-1) & 1 \\ 5 & -(-1) \end{bmatrix}$

$$\begin{bmatrix} 5 & 1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

solving $(A - \lambda I)x = \vec{0}$

$$= \begin{bmatrix} 5(1) + 1(-5) \\ 5(1) + 1(-5) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Example: Let $T(\mathbf{x}) = \begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \mathbf{x}$.

Can solve $\begin{bmatrix} 5 & 1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$(A - \lambda I)\vec{x} = \vec{0}$

by inspection

or use row ops

$\left[\begin{array}{cc|c} 5 & 1 & 0 \\ 5 & 1 & 0 \end{array}\right]$

Example: Let $T(\mathbf{x}) = \begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \mathbf{x}$.

$\lambda = 5$: $A - 5I = \begin{bmatrix} 4-5 & 1 \\ 5 & 0-5 \end{bmatrix}$

$$\begin{bmatrix} -1 & 1 \\ 5 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ e. vector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$

espace is $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 100{,}000 \\ 100{,}000 \end{bmatrix}$

Example: Let $T(\mathbf{x}) = \begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \mathbf{x}$.

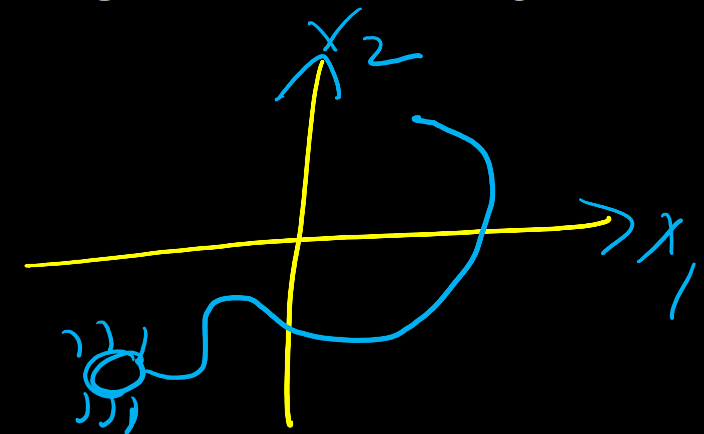
The set of e. vectors w/ e. value 5 is

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_1 \end{bmatrix}$$

where $c_1 \neq 0$

Chapter 7: Systems of Linear DE

Suppose an object moves in the 2D plane (the x_1, x_2 plane) so that it is at the point $(x_1(t), x_2(t))$ at time t. Suppose the object's velocity is given by



$$x_1'(t) = 4x_1 + x_2,$$
$$x_2'(t) = 5x_1$$

Write in matrix form:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}' = \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

To solve
find
e. value
e. vectors, per Wed class

$$\vec{x}' = \begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix} \vec{x}$$

Homogeneous vs non-homogeneous system of linear DE:

Homogeneous

$$\vec{x}' = A\vec{x}$$

$$\vec{x}' - A\vec{x} = 0 \quad \leftarrow \text{homog}$$

Non homog

$$\vec{x}' = A\vec{x} + \vec{b}$$

$$\vec{x}' - A\vec{x} = \vec{b} \quad \leftarrow \text{Non hom}$$

7.1: Transforming an n^{th} order linear DE into a system of n first order linear DEs.

with n unknows

Ex: $\boxed{y'' - 5y' + 6y = 0}$

2 unknowns: x_1 and x_2

Let $y = x_1$

$$y' = x_1' = x_2$$

$$y'' = x_2' = -6x_1 + 5x_2$$

system of 2 linear first order eqn

$$y'' = 5y' - 6y = -6y + 5y'$$

7.1: Transforming an n^{th} order linear DE into a system of n first order linear DEs.

2 unknowns w/ n unknowns

Ex: $y'' - 5y' + 6y = 0$, $\boxed{y(0) = 1, \quad y'(0) = 2}$

$y'' = -6y + 5y'$

$x_1(0) = 1$

$x_2(0) = 2$

Let $y = x_1$

$$\boxed{\begin{aligned} y' &= x_1' = x_2 \\ y'' &= x_2' = -6x_1 + 5x_2 \end{aligned}}$$

$$\bar{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\vec{x}' = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

7.1: Transforming an n^{th} order linear DE into a system of n first order linear DEs.

non homog

Ex: $y'' - 5y' + 6y = \sin(t)$

$y'' = -6y + 5y' + \sin(t)$

$y = x_1$

$y' = x_1' = x_2$

$y'' = x_2' = -6x_1 + 5x_2 + \sin(t)$

in matrix form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \sin(t) \end{bmatrix}$$

7.1: Transforming an n^{th} order linear DE into a system of n first order linear DEs.

Ex: $y'''' - 5y'' + 6y = \sin(t) \Rightarrow y'''' = -6y + 5y'' + \sin(t)$

$$y = x_1$$

$$y' = x_1' = x_2$$

$$y'' = x_2' = x_3$$

$$y''' = x_3' = x_4$$

$$y'''' = x_4' = -6x_1 + 5x_3 + \sin(t)$$

see Wed for Matrix format