1.1: direction field = slope field
= graph of $\frac{dy}{dt}$ in t, y-plane.

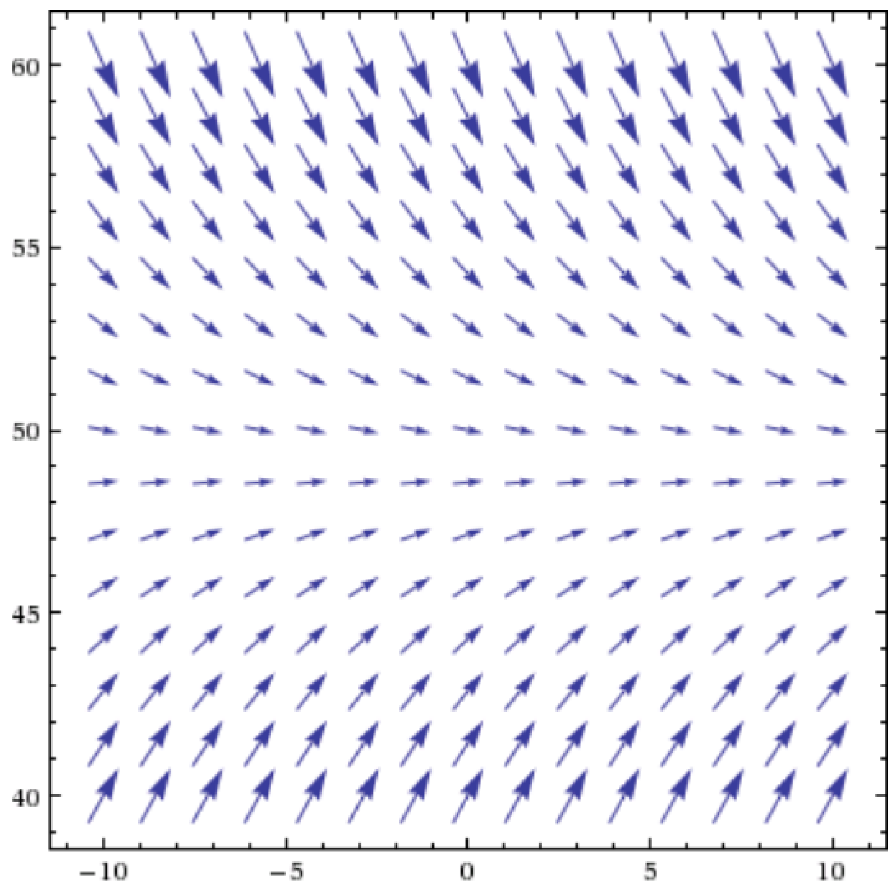
* Can use slope field to determine behavior of y including as $t \to \pm\infty$.

* Equilibrium Solution = constant solution

Most differential equations do not have an equilibrium solution.

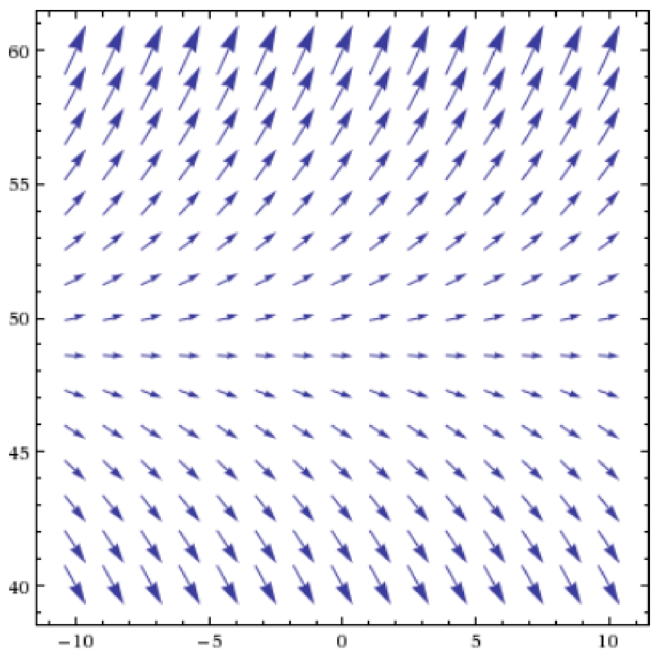
1.1: Examples of differentiable equation:

1.) Ball example: $F = ma = m\frac{dv}{dt} = mg - \gamma v$



2.) Mouse population increases at a rate proportional to the current population:

Model : $\frac{dp}{dt} = rp - k$
where $p(t) =$ mouse population at time t,
$r =$ growth rate or rate constant,
$k =$ predation rate $= \#$ mice killed per unit time.



3.) Continuous compounding $\frac{dS}{dt} = rS + k$
where $S(t) =$ amount of money at time t,
$r =$ interest rate,
$k =$ constant deposit rate

*************Existence of a solution**************

*************Uniqueness of solution**************

Knowing whether or not a solution exists, and if it exists whether or not that solution is unique is extremely useful.

Show that for some value of r, $\quad y = e^{rt}$ is a soln to the 1^{rst} order linear homogeneous eq'n $2y' - 6y = 0$.

To show something is a solution, plug it in:

Claim: $y = e^{rt}$ is a soln to $2y' - 6y = 0$ for some r.

$y = e^{rt}$ implies $y' = re^{rt}$. Plug into $2y' - 6y = 0$:

$2re^{rt} + 6e^{rt} = 0$ implies $2r - 6 = 0$ implies $r = 3$

Thus $y = e^{3t}$ is a solution to $2y' - 6y = 0$.

Claim: $y = Ce^{3t}$ is a solution to $2y' - 6y = 0$.

$$2y' - 6y = 2(Ce^{3t})' - 6(Ce^{3t}) = 2C(e^{3t})' - 6C(e^{3t})$$
$$= C[2(e^{3t})' - 6(e^{3t})] = C(0) = 0.$$

IVP: Solve $2y' - 6y = 0$, $y(0) = 4$

From previous slide, we know $y = Ce^{3t}$ is a solution to $2y' - 6y = 0$.

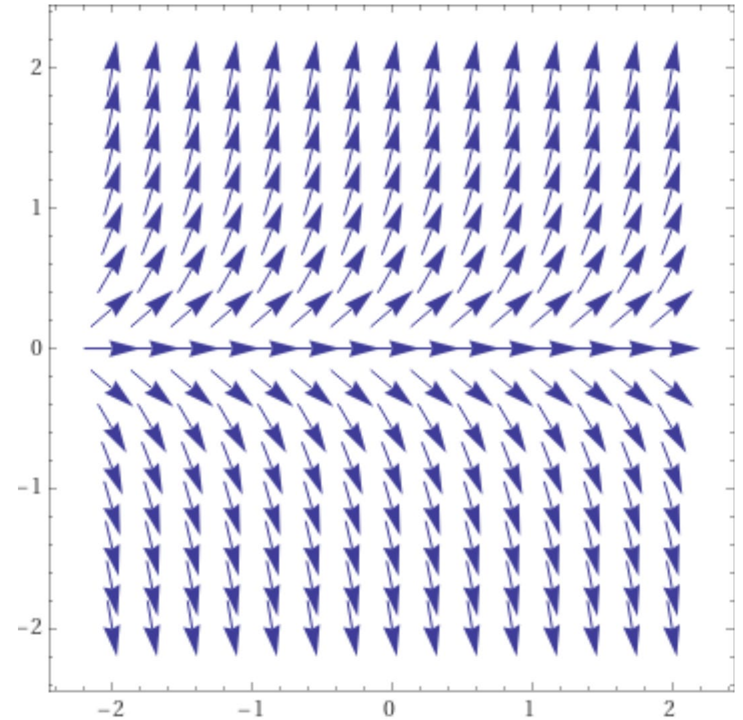
Plug in initial value to find C:

If $y(0) = 4$, then $4 = Ce^{3(0)}$ implies $C = 4$.

Thus by existence and uniqueness thm, $y = 4e^{3t}$ is the unique solution to IVP: $2y' + 6y = 0$, $y(0) = 4$.

Initial value: A chosen point (t_0, y_0) through which a solution must pass.

I.e., (t_0, y_0) lies on the graph of the solution that satisfies this initial value.



https://www.wolframalpha.com/input/?i=slope%20field

Initial value problem (IVP): A differential equation where initial value is specified.

An initial value problem can have 0, 1, or multiple equilibrium solutions (finite or infinite).

http://www.wolframalpha.com

slope field

Web Apps | Examples | Random

Assuming "slope field" refers to a computation | Use as referring to a mathematical definition instead

{1, (ln(x) + y)}/sqrt(1 + (ln(x) + y)^2)

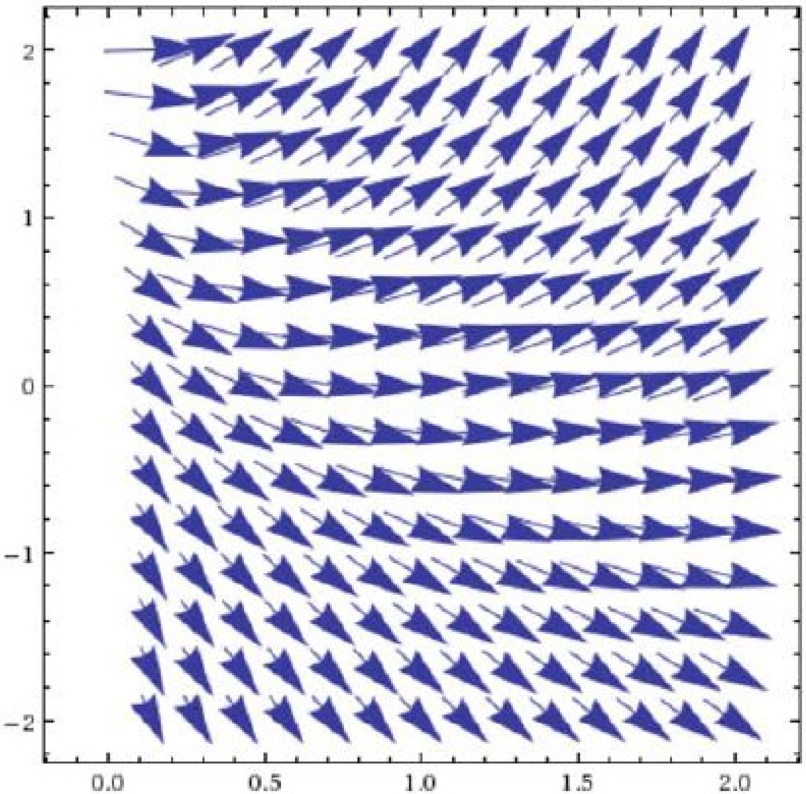
- vector field: {1, (ln(x) + y)}/sqrt(1
- variable 1: x
- lower limit 1: 0
- upper limit 1: 2
- variable 2: y
- lower limit 2: -2
- upper limit 2: 2

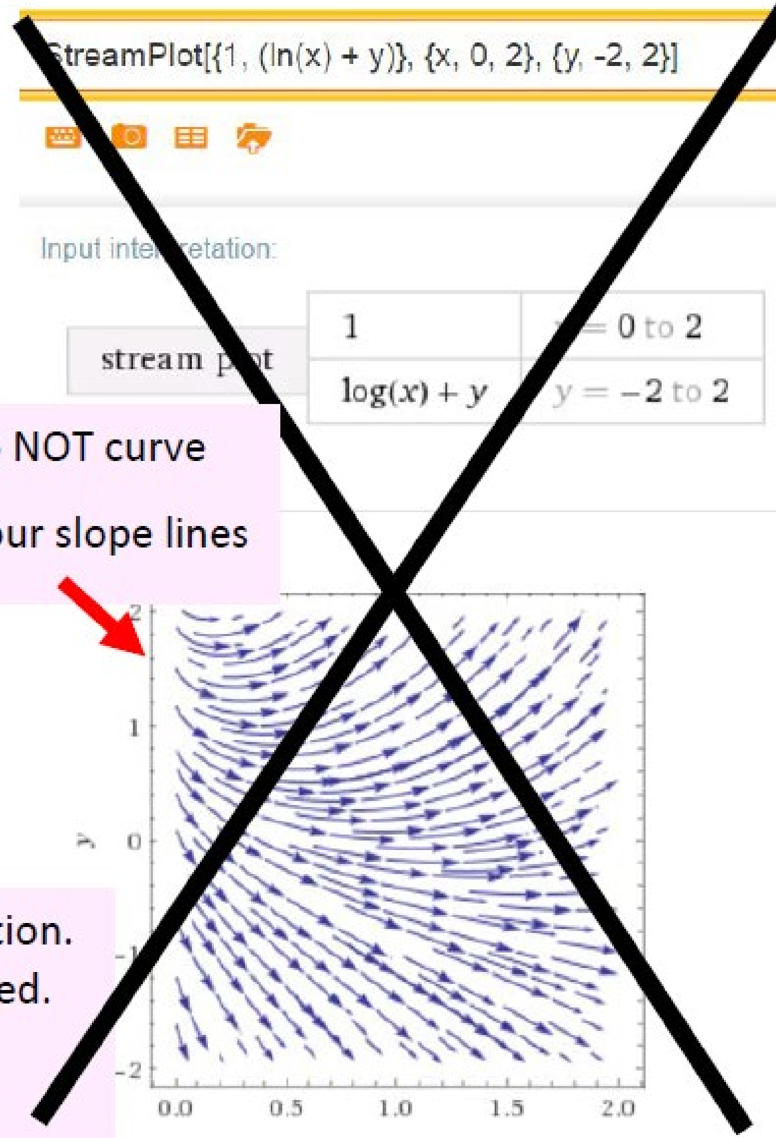
Input:

$$\text{VectorPlot}\left[\frac{\{1, \log(x) + y\}}{\sqrt{1 + (\log(x) + y)^2}}, \{x, 0, 2\}, \{y, -2, 2\}\right]$$

log(x) is the natural logarithm

Result:





Do NOT curve your slope lines

Slope lines are small portions of lines tangent to a solution. Thus slope lines must be straight. They cannot be curved.

Arrows are optional

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Ch 2.2: Separable Equations

- In this section we examine a subclass of linear and nonlinear first order equations. Consider the first order equation

$$\frac{dy}{dx} = f(x, y)$$

- We can rewrite this in the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

- For example, let $M(x,y) = -f(x,y)$ and $N(x,y) = 1$. There may be other ways as well. In differential form,

$$M(x, y)dx + N(x, y)dy = 0$$

- If M is a function of x only and N is a function of y only, then

$$M(x)dx + N(y)dy = 0$$

- In this case, the equation is called **separable**.

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Example 1: Solving a Separable Equation

- Solve the following first order nonlinear equation:

$$\frac{dy}{dx} = \frac{x^2 + 1}{y^2 - 1}$$

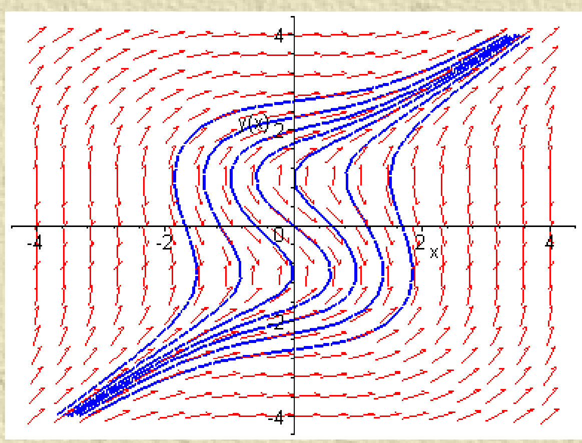
- Separating variables, and using calculus, we obtain

$$\left(y^2 - 1\right)dy = \left(x^2 + 1\right)dx$$

$$\int \left(y^2 - 1\right)dy = \int \left(x^2 + 1\right)dx$$

$$\frac{1}{3}y^3 - y = \frac{1}{3}x^3 + x + C$$

$$y^3 - 3y = x^3 + 3x + C$$



- The equation above defines the solution *y* implicitly. A graph showing the direction field and implicit plots of several integral curves for the differential equation is given above.

Example 2: Implicit and Explicit Solutions (1 of 4)

- Solve the following first order nonlinear equation:

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}$$

- Separating variables and using calculus, we obtain

$$2(y-1)dy = \left(3x^2 + 4x + 2\right)dx$$

$$2\int (y-1)dy = \int \left(3x^2 + 4x + 2\right)dx$$

$$y^2 - 2y = x^3 + 2x^2 + 2x + C$$

- The equation above defines the solution y implicitly. An explicit expression for the solution can be found in this case:

$$y^2 - 2y - \left(x^3 + 2x^2 + 2x + C\right) = 0 \quad \Rightarrow \quad y = \frac{2 \pm \sqrt{4 + 4\left(x^3 + 2x^2 + 2x + C\right)}}{2}$$

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + C}$$

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Example 2: Initial Value Problem (2 of 4)

- Suppose we seek a solution satisfying $y(0) = -1$. Using the implicit expression of y, we obtain

$$y^2 - 2y = x^3 + 2x^2 + 2x + C$$

$$(-1)^2 - 2(-1) = C \quad \Rightarrow \quad C = 3$$

- Thus the implicit equation defining y is

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3$$

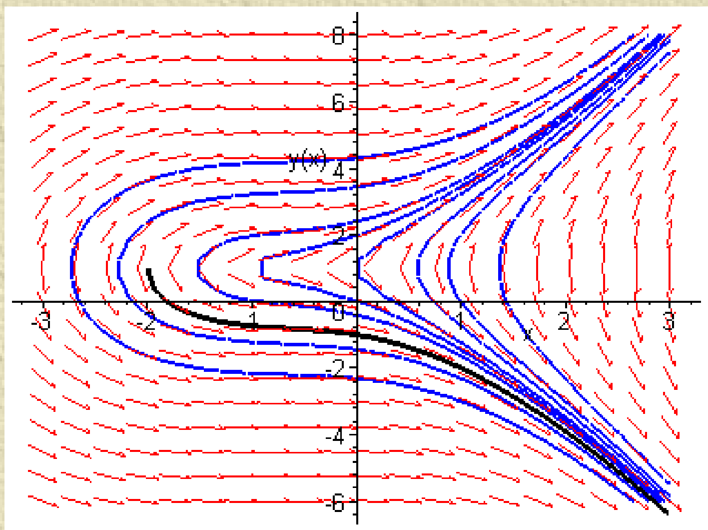
- Using explicit expression of y,

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + C}$$

$$-1 = 1 \pm \sqrt{C} \quad \Rightarrow \quad C = 4$$

- It follows that

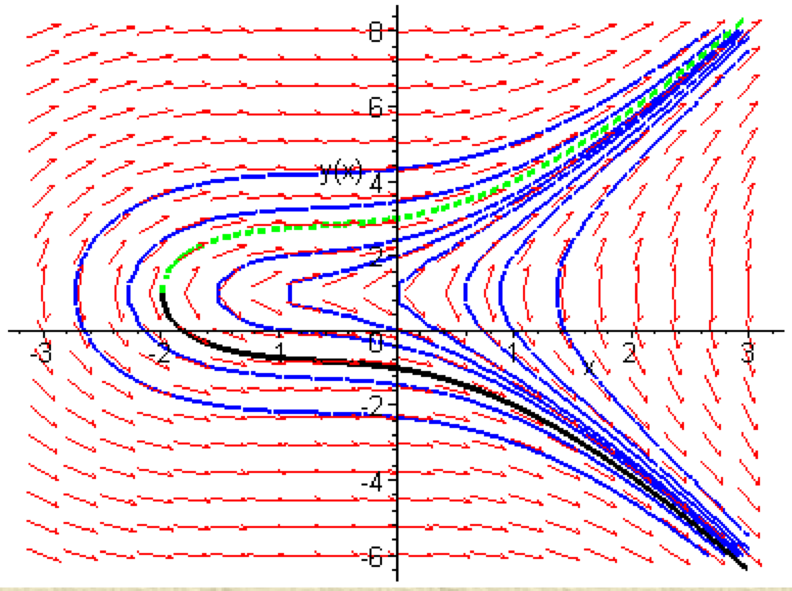
$$y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}$$



Example 2: Initial Condition $y(0) = 3$ (3 of 4)

- Note that if initial condition is $y(0) = 3$, then we choose the positive sign, instead of negative sign, on square root term:

$$y = 1 + \sqrt{x^3 + 2x^2 + 2x + 4}$$



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Example 2: Domain (4 of 4)

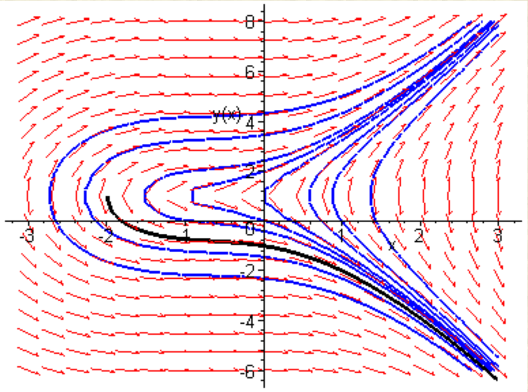
- Thus the solutions to the initial value problem

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \quad y(0) = -1$$

are given by

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3 \quad \text{(implicit)}$$

$$y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4} \quad \text{(explicit)}$$



- From explicit representation of y, it follows that

$$y = 1 - \sqrt{x^2(x+2) + 2(x+2)} = 1 - \sqrt{(x+2)(x^2+2)}$$

and hence domain of y is $(-2, \infty)$. Note $x = -2$ yields $y = 1$, which makes denominator of dy/dx zero (vertical tangent).

- Conversely, domain of y can be estimated by locating vertical tangents on graph (useful for implicitly defined solutions).

Example 3: Implicit Solution of Initial Value Problem (1 of 2)

- Consider the following initial value problem:

$$y' = \frac{y\cos x}{1+3y^3}, \quad y(0) = 1$$

- Separating variables and using calculus, we obtain

$$\frac{1+3y^3}{y}dy = \cos x dx$$

$$\int\left(\frac{1}{y} + 3y^2\right)dy = \int \cos x dx$$

$$\ln|y| + y^3 = \sin x + C$$

- Using the initial condition, it follows that

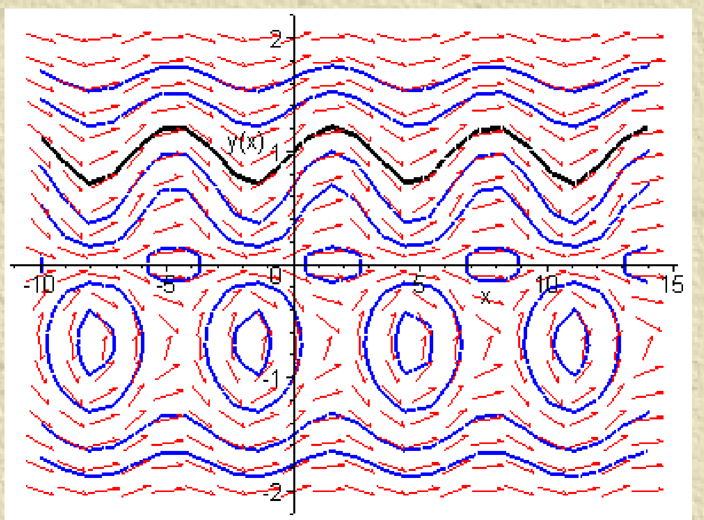
$$\ln y + y^3 = \sin x + 1$$

Example 3: Graph of Solutions (2 of 2)

- Thus

$$y' = \frac{y\cos x}{1+3y^3}, \quad y(0)=1 \;\Rightarrow\; \ln y + y^3 = \sin x + 1$$

- The graph of this solution (black), along with the graphs of the direction field and several integral curves (blue) for this differential equation, is given below.



Section 1.3: Classification of differential equations: Some additional terminology

ODE (ordinary differential equation): single independent variable

Ex: $\frac{dy}{dt} = ay + b$

PDE (partial differential equation): several independent variables

Ex: $\frac{\partial xy}{\partial x} = \frac{\partial xy}{\partial y}$

order of differential eq'n $=$ order of highest derivative

example of order n: $y^{(n)} = f(t, y, ..., y^{(n-1)})$

Ex 1: $y^{(6)} = y^9 t^7 y^v y' - y''' y^{iv}$ has order ___________

Ex 2: $y^6 = y^9 t^7 y^v y' - y''' y^{iv}$ has order ___________

Linear vs Non-linear

Linear: $a_0 y^{(n)} + ... + a_{n-1} y' + a_n y = g(t)$

where a_i's are functions of t

Note for this linear equation, the left hand side is a **linear combination** of the derivatives of y (denoted by $y^{(k)}, k = 0, ..., n$) where the coefficient of $y^{(k)}$ is a function of t (denoted $a_k(t)$).

Linear: $a_0(t)y^{(n)} + ... + a_{n-1}(t)y' + a_n(t)y = g(t)$

Determine if linear or non-linear:

Ex 1: $ty'' - t^3y' - 3y = sin(t)$ is ______________

Ex 2: $2y'' - 3y' - 3y^2 = 0$ is ________________

Ex 3:
$ty - t^3y'' - ln(cos(t)y^{vi} + \sqrt{\frac{ln(cost)}{\sqrt{t+1}}} = y^v sin(t)$ is