Exams are graded

Please see comments in uploaded file and check my addition, etc.

Please let me Know if you have any questions.

$\left(\quad\right)\frac{3}{2} = \boxed{\text{grade}}$

Consequence 2:

If ψ_1 is a solution to $\overbrace{af'' + bf' + cf}^{\text{LHS}} = h$

and ψ_2 is a solution to $af'' + bf' + cf = k$,

then $3\psi_1 + 5\psi_2$ is a solution to $af'' + bf' + cf = 3h + 5k$,

Since ψ_1 is a solution to $af'' + bf' + cf = h$, $L(\psi_1) = h$. RHS

Since ψ_2 is a solution to $af'' + bf' + cf = k$, $L(\psi_2) = k$.

$L(f) = af'' + bf' + cf$ is a linear fn

Hence $L(3\psi_1 + 5\psi_2) = 3L(\psi_1) + 5L(\psi_2)$

$$= 3h + 5k.$$

Thus $3\psi_1 + 5\psi_2$ is also a solution to

$$\underbrace{af'' + bf' + cf}_{\text{LHS}} = 3h + 5k$$

3.5

Thm: Suppose $c_1\phi_1(t) + c_2\phi_2(t)$ is a general solution to
$$ay'' + by' + cy = 0,$$ ← homog

If ψ is a solution to
$$ay'' + by' + cy = g(t) \ [*],$$ ← non homog

hyp

Then $\psi + c_1\phi_1(t) + c_2\phi_2(t)$ is also a solution to [*]. concl

hyp stated

Moreover if γ is also a solution to [*], then there exist constants c_1, c_2 such that
$$\gamma = \psi + c_1\phi_1(t) + c_2\phi_2(t)$$

conclusion

Or in other words, $\boxed{\psi + c_1\phi_1(t) + c_2\phi_2(t)}$ is a general solution to [*].

all soln's look like this

main conclusion

Proof:

Define $L(f) = af'' + bf' + cf$. ← LHS of DE

Recall L is a linear function.

hyp

Since $c_1\phi_1(t) + c_2\phi_2(t)$ is a solution to the differential equation, $ay'' + by' + cy = 0$,

$$L(c_1\phi_1 + c_2\phi_2) = 0$$ ← plug into LHS of $ay''+by'+cy=0$ get RHS since soln

hyp

Since ψ is a solution to $ay'' + by' + cy = g(t)$,

$$L(\psi) = g(t)$$

$ay''+by'+cg$ $c_1 L(\phi_1), c_2L(\phi_2)$

We will now show that $\psi + c_1\phi_1(t) + c_2\phi_2(t)$ is also a solution to [*]. ← non homog DE

$$L(\psi + c_1\phi_1 + c_2\phi_2) = L(\psi) + L(c_1\phi_1 + c_2\phi_2)$$
$$= g(t) + 0 = g(t)$$

Claim: ψ $c_1\phi_1(t) + c_2\phi_2(t)$ $\psi + c_1\phi_1 + c_2\phi_2$ is a general solution to

$$ay'' + by' + cy = 0,$$

Since γ a solution to $ay'' + by' + cy = g(t)$, ← hypothesis

$$L(\gamma) = g(t)$$

We will first show that $\gamma - \psi$ is a solution to the differential equation $ay'' + by' + cy = 0$. ← homog

$$L(\gamma - \psi) = L(\gamma) - L(\psi) = g(t) - g(t) = 0$$

Since $\gamma - \psi$ is a solution to $ay'' + by' + cy = 0$ and

$c_1\phi_1(t) + c_2\phi_2(t)$ is a general solution to

$$ay'' + by' + cy = 0,$$ ← hypothesis

there exist constants c_1, c_2 such that

$$\gamma - \psi^{+\psi} = c_1\phi_1 + c_2\phi_2 {}^{+\psi}$$

Thus $\gamma = \psi + c_1\phi_1(t) + c_2\phi_2(t)$. ← conclusion □

Thm:
Suppose f_1 is a a solution to $ay'' + by' + cy = g_1(t)$ and f_2 is a a solution to $ay'' + by' + cy = g_2(t)$, then $f_1 + f_2$ is a solution to $\boxed{ay'' + by' + cy = g_1(t) + g_2(t)}$

Proof: Let $L(f) = af'' + bf' + cf$.

Since f_1 is a solution to $ay'' + by' + cy = g_1(t)$, $(*)$

$L(f_1) = g_1$

Since f_2 is a solution to $ay'' + by' + cy = g_2(t)$, $(**)$

$L(f_2) = g_2$

We will now show that $f_1 + f_2$ is a solution to $ay'' + by' + cy = g_1(t) + g_2(t)$. $(***)$

$L(f_1 + f_2) = L(f_1) + L(f_2) = g_1 + g_2$

$\Rightarrow f_1 + f_2$ is a soln to $(***)$

Sidenote: The proofs above work even if a, b, c are functions of t instead of constants.

3.5

$\boxed{\text{To solve } ay'' + by' + cy = g_1(t) + g_2(t) + ...g_n(t) \text{ [**]}}$

1.) Find the general solution to $ay'' + by' + cy = 0$:

$$c_1\phi_1 + c_2\phi_2$$

solve homog

2.) For each g_i, find a solution to $ay'' + by' + cy = g_i$:

$$\psi_i$$

This includes plugging guessed solution ψ_i into $ay'' + by' + cy = g_i$.

The general solution to [**] is

$$\boxed{y = c_1\phi_1 + c_2\phi_2 + \psi_1 + \psi_2 + ...\psi_n}$$

3.) If initial value problem:

Once general solution is known, can solve initial value problem (i.e., use initial conditions to find c_1, c_2).

plug in $\left.\begin{matrix} y(t_0) = y_0 \\ y'(t_0) = y_1 \end{matrix}\right\}$ solve for c_1 & c_2

plug into general NON-homog soln

Solve $y'' - 4y' - 5y = 4sin(3t)$, $y(0) = 6$, $y'(0) = 7$. wed ex

1.) **First solve homogeneous equation:**

Find the general solution to $y'' - 4y' - 5y = 0$:

Guess $y = e^{rt}$ for HOMOGENEOUS equation:

$y' = re^{rt}$, $y' = r^2e^{rt}$

$y'' - 4y' - 5y = 0$

$r^2e^{rt} - 4re^{rt} - 5e^{rt} = 0$

$e^{rt}(r^2 - 4r - 5) = 0$

$e^{rt} \neq 0$, thus can divide both sides by e^{rt}:

$$r^2 - 4r - 5 = 0$$

$(r + 1)(r - 5) = 0$. Thus $r = -1, 5$.

Thus $y = e^{-t}$ and $y = e^{5t}$ are both solutions to LINEAR HOMOGENEOUS equation.

Thus the general solution to the 2nd order LINEAR HOMOGENEOUS equation is

$$y = c_1e^{-t} + c_2e^{5t}$$

Guessed $y = A\sin 3t + B\cos(3t)$

Method of undetermined coef

2.) **Find one solution to non-homogeneous eq'n:**

Find a solution to $ay'' + by' + cy = 4sin(3t)$:

9, (+)

Guess $y = Asin(3t) + Bcos(3t)$

$$y' = 3Acos(3t) - 3Bsin(3t)$$

$$y'' = -9Asin(3t) - 9Bcos(3t)$$

3.5 educated guess

$y'' - 4y' - 5y = 4sin(3t)$

Solve for undetermined coefficients A & B by plugging in our guess

$$\begin{array}{llcl}
y'' & -9Asin(3t) & - & 9Bcos(3t) \\
-4y' & 12Bsin(3t) & - & 12Acos(3t) \\
-5y & -5Asin(3t) & - & 5cos(3t) \\
\hline
 & (12B - 14A)sin(3t) & - & (-14B - 12A)cos(3t) \quad = \quad 4sin(3t)
\end{array}$$

RHS

Since $\{sin(3t), cos(3t)\}$ is a linearly independent set:

$= 4\sin 3t + 0\cos 3t$

$12B - 14A = 4$ and $-14B - 12A = 0$

0 cosine's

Thus $A = -\frac{14}{12}B = -\frac{7}{6}B$ and

4 sines

$12B - 14(-\frac{7}{6}B) = 12B + 7(\frac{7}{3}B) = \frac{36+49}{3}B = \frac{85}{3}B = 4$

Thus $B = 4(\frac{3}{85}) = \frac{12}{85}$ and $A = -\frac{7}{6}B = -\frac{7}{6}(\frac{12}{85}) = -\frac{14}{85}$

Thus $y = (-\frac{14}{85})sin(3t) + \frac{12}{85}cos(3t)$ is one solution to the nonhomogeneous equation.

Thus the general solution to the 2nd order linear nonhomogeneous

$y = (c_1 e^{-t} + c_2 e^{5t}) +$

homog non homog

equation is

$$\boxed{y = c_1 e^{-t} + c_2 e^{5t} - (\tfrac{14}{85}) sin(3t) + \tfrac{12}{85} cos(3t)}$$

3.) **If initial value problem:**

Once general solution is known, can solve initial value problem (i.e., use initial conditions to find c_1, c_2).

to non homog

NOTE: you must know the GENERAL solution to the ODE BEFORE you can solve for the initial values. The homogeneous solution and the one nonhomogeneous solution found in steps 1 and 2 above do NOT need to separately satisfy the initial values.

Solve $y'' - 4y' - 5y = 4sin(3t)$, $y(0) = 6$, $y'(0) = 7$.

General solution: $y = c_1 e^{-t} + c_2 e^{5t} - (\frac{14}{85}) sin(3t) + \frac{12}{85} cos(3t)$

Thus $y' = -c_1 e^{-t} + 5c_2 e^{5t} - (\frac{42}{85}) cos(3t) - \frac{36}{85} sin(3t)$

$y(0) = 6$: $\quad 6 = c_1 + c_2 + \frac{12}{85} \qquad \frac{498}{85} = c_1 + c_2$

$y'(0) = 7$: $\quad 7 = -c_1 + 5c_2 - \frac{42}{85} \qquad \frac{637}{85} = -c_1 + 5c_2$

$6c_2 = \frac{498+637}{85} = \frac{1135}{85} = \frac{227}{17}$. Thus $c_2 = \frac{227}{102}$.

$c_1 = \frac{498}{85} - c_2 = \frac{498}{85} - \frac{227}{102} = \frac{2988-1135}{510} = \frac{1853}{510} = \frac{109}{30}$

Thus $y = (\frac{109}{30})e^{-t} + (\frac{227}{102})e^{5t} - (\frac{14}{85}) sin(3t) + \frac{12}{85} cos(3t)$.

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check

non homog

$(\quad)'' - 4(\quad) - 5(\quad)$

Partial Check: $y(0) = (\frac{109}{30}) + (\frac{227}{102}) + \frac{12}{85} = 6.$

$$y'(0) = -\frac{109}{30} + 5(\frac{227}{102}) - \frac{42}{85} = 7.$$

satisfies initial values ✓

$(e^{-t})'' - 4(e^{-t})' - 5(e^{-t}) = 0$ and $(e^{5t})'' - 4(e^{5t})' - 5(e^{5t}) = 0$

easy to check homog part

Checking non homog part is more work

what/

3.5: How do you guess

Examples: Find a suitable form for ψ for the following differential equations:

Step 1: Solve homog $\oslash\oslash$

$r^2-4r-5=0$ $\oslash$

$\Rightarrow y=e^{5t}$, $y=e^{-t}$ } homog soln

1.) $y''-4y'-5y=4e^{2t}$

Guess $\psi = Ae^{2t}$ ← plug in and solve for A

2.) $y''-4y'-5y=t^2-2t+1$ ← 2nd deg polynomial

Guess: $\psi(t)=At^2+Bt+C$

3.) $y''-4y'-5y=4sin(3t)$ ← Ex from wed

Guess

$\psi(t)=A\sin(3t)+B\cos(3t)$

"obvious" based on trial & error

4.) $y''-5y=4sin(3t)$

Guess $\psi(t)=A\sin(3t)$

Don't need $B\cos(3t)$ term since no y' term

5.) $y''-4y'=t^2-2t+1$ ← 3 equations

Guess: $\psi = t(At^2+Bt+C)$

6.) $y''-4y'-5y=4(t^2-2t-1)e^{2t}$

no y term

$y'' \stackrel{?}{=} 4y'$

7.) $y'' - 4y' - 5y = 4sin(3t)e^{2t}$

8.) $y'' - 4y' - 5y = 4(t^2 - 2t - 1)sin(3t)e^{2t}$

9.) $y'' - 4y' - 5y = 4sin(3t) + 4sin(3t)e^{2t}$

10.) $y'' - 4y' - 5y$

$$= 4sin(3t)e^{2t} + 4(t^2 - 2t - 1)e^{2t} + t^2 - 2t - 1$$

11.) $y'' - 4y' - 5y = 4sin(3t) + 5cos(3t)$

12.) $y'' - 4y' - 5y = 4e^{-t}$ ← homog

Plug $\phi = Ae^{-t}$ $\Rightarrow$ $(\ )'' - 4(\ )' - 5(\ ) = 0$

Guess $\psi = Ate^{-t}$