Sunday, September 13, 2020 3:02 PM

New office hour: Monday 4:30 - 5:30pm

2.8: We will outline the proof to Thm 2.4.2

Note: The proof will be constructive. That is

We will create functions that approximate the solution to the IVP

We will use the **Method of Successive Approximation** (also called Picard's iteration method) to create functions $y = \phi_n(t)$ such that

$\phi_1, \phi_2, \phi_3, \ldots \Longrightarrow \phi$

$$\underbrace{\phi(t) = \lim_{n\to\infty} \phi_n(t)}$$

a solution to IVP: $y' = f(t, y)$, $y(0) = 0$.

IVP: $y' = f(t, y)$, $y(t_0) = y_0$. $\longrightarrow z' = g(s, z)$, $z(0) = 0$

Can translate IVP to move initial value to the origin and translate back after solving:

z

y

$\boxed{\text{Let } s = t - t_0}$

y_0

y_0 t_0

Let $s = t - t_0$

Let $z = y - y_0$

s

So for 2.8 initial value will be $y(0) = 0$ to make algebra easier

$z = y - y_0 \Rightarrow z' = y'$

Thm 2.8.1 is translated to origin version of Thm 2.4.2:

Thm 2.8.1: Suppose the functions

$$z = f(t, y) \text{ and } z = \frac{\partial f}{\partial y}(t, y)$$

are continuous for all t in $(-a, a) \times (-c, c)$, then there exists an interval $(-h, h) \subset (-a, a)$ such that there exists a unique function $y = \phi(t)$ defined on $(-h, h)$ that satisfies the following initial value problem:

$$\boxed{y' = f(t, y), \quad y(0) = 0.}$$

t c $-a$ a $-c$

initial value $(0, 0)$

Thm 2.8.1 $\Leftrightarrow$ Thm 2.4.2

$(\Rightarrow)$ Proof $\left[\begin{array}{l} s \leftrightarrow t - t_0 \\ z \leftrightarrow y - y_0 \end{array}\right]$

$\Leftarrow$ Let $t_0 = 0$, $y_0 = 0$ $\square$

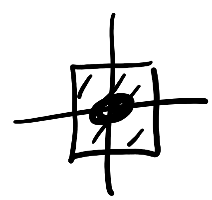
Proof idea / outline / Scratch work

Given:

$$\boxed{y' = f(t,y),\ y(0) = 0} \quad (\text{Eqn } ✪)$$

$z = f(t,y)$ and $z = \frac{\partial f}{\partial y}(t,y)$

are both continuous near $(0,0)$



Claim: $\exists!$ soln to Eqn ✪

Suppose $y = \phi(t)$ is a soln to Eqn ✪

$y = \phi(t)$ is a soln to $y' = f(t,y)$, $y(0) = 0$

$$\Longleftrightarrow \quad \phi'(t) = f(t, \phi(t))$$
$$\phi(0) = 0$$

$$\Longleftrightarrow \quad \int_0^t \phi'(s)\,ds = \int_0^t f(s, \phi(s))\,ds$$
and $\phi(0) = 0$

$$\overset{\text{FTC}}{\Longleftrightarrow} \quad \phi(s)\Big|_0^t = \int_0^t f(s, \phi(s))\,ds$$
and $\phi(0) = 0$

$$\Longleftrightarrow \quad \phi(t) - \phi(0) = \int_0^t f(s, \phi(s))\,ds$$
and $\underline{\phi(0) = 0}$

$$\Longleftrightarrow \quad \phi(t) = \int_0^t f(s, \phi(s))\,ds$$

Not: RH integral exists since f cont assuming $\underline{\phi \text{ exists}}$

↑ problem

(this is what we are trying to prove

Claim

$$\boxed{\phi(t) = \int_0^t f(s, \phi(s))\, ds}$$

is a soln to eqn A

Finding ϕ in terms of ϕ does not give us ϕ

Thus will create a sequence of fns using this formula

Let $\phi_0(t) = 0$ (or any continuous fn that you like)

Looking for soln to

$\phi(t) = \int y' = \int_0^t f(s, y(s))\, \phi$ $y' = f(t, y)$

Let $\phi_1(t) = \int_0^t f(s, \phi_0(s))\, ds$

Let $\phi_2(t) = \int_0^t f(s, \phi_1(s))\,ds$

$\vdots$

Let $\boxed{\phi_n(t) = \int_0^t f(s, \phi_{n-1}(s))\,ds}$

Claim $\lim_{n\to\infty} \phi_n(t) = \phi(t)$

where $\phi(t)$ is a $\overset{!}{\text{soln}}$ to Eqn ⊛

$y' = f(t, y)$

$y(0) = 0$

The def'n of ϕ_n is an inductive defn

to find ϕ_n, need to know ϕ_{n-1}

" " ϕ_{n-1} " ϕ_{n-2}

$\vdots$

ϕ_1 " $\underbrace{\phi_0}_{\text{base case}}$

ϕ_n is defined recursively

Example: $y' = t + 2y = f(t,y)$
$y(0) = 0$

$$f(t,y) = t + 2y$$

$$\phi_n(t) = \int_0^t f(\underset{t}{s}, \underset{y}{\phi_{n-1}(s)})\,ds$$

$$\text{Let } \phi_0(t) = 0$$

$$\phi_1(t) = \int_0^t f(s, \phi_0(s))\,ds$$

$$= \int_0^t f(s, 0)\,ds$$

$f(t,y) = t+2y$

$$= \int_0^t (s + 2(0))\,ds$$

$$= \int_0^t s\,ds = \frac{s^2}{2}\Big|_0^t$$

$$= \frac{t^2}{2}$$

$$2$$

$$\phi_2(t) = \int_0^t f(s, \phi_1(s))\,ds$$

$$f(t,y)$$
$$= t + 2y$$

$$= \int_0^t f\left(s, \frac{s^2}{2}\right) ds$$

$$= \int_0^t \left(s + \cancel{2}\left(\frac{s^2}{\cancel{2}}\right)\right) ds$$

$$= \int_0^t (s + s^2)\,ds$$

$$= \frac{s^2}{2} + \frac{s^3}{3}\Big|_0^t$$

$$= \frac{t^2}{2} + \frac{t^3}{3} - (0 + 0)$$

$$\phi_3(t) = \int_0^t f(s, \underbrace{\phi_2(s)}_{y \text{ approx by } \phi_2})\,ds$$

$$\phi_3(t) = \int_0^t f(s, \phi_2(s))\,ds$$

$$= \int_0^t f\left(s, \frac{s^2}{2} + \frac{s^3}{3}\right) ds$$

$$= \int_0^t \left[s + 2\left(\frac{s^2}{2} + \frac{s^3}{3}\right)\right] ds$$

$$= \int_0^t \left[s + s^2 + \frac{2s^3}{3}\right] ds$$

$$= \frac{s^2}{2} + \frac{s^3}{3} + \frac{\cancel{2}s^4}{\underset{2}{\cancel{4}} \cdot 3}\Bigg|_0^t$$

$$= \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6} - [0+0+0]$$

$$\phi_0 = 0$$

$$\phi_0 = 0$$

$$\phi_1 = \frac{t^2}{2}$$

$$\phi_2 = \frac{t^2}{2} + \frac{t^3}{3}$$

$$\boxed{\phi_n(t) = \sum_{K=1}^{n} \left(\frac{t^{K+1}}{?}\right)}$$

$$\phi_3 = \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6}$$

$\phi_n(t) = \ ?$ ← Find formula

$$\phi_n(t) = \sum_{K=1}^{n} \frac{t^{K+1}}{\sim} = \sum_{j=2}^{n+1} \frac{t^j}{\ } = \sum_{j=0}$$

$$= \frac{t^{1+1}}{\sim} + \frac{t^{2+1}}{\sim} + \frac{t^{3+1}}{\sim} + \dots + \frac{t^{n+1}}{\sim}$$

$$\phi_3(t) = \frac{t^2}{\sim} + \frac{t^3}{\sim} + \frac{t^4}{\sim}$$

Scratch

$$\int t = \frac{t^2}{2} \rightarrow \int \frac{t^2}{2} = \frac{t^3}{3 \cdot 2}$$

$$\int t^2 = \frac{t^3}{3} \qquad \int \frac{t^3}{3 \cdot 2} = \frac{t^4}{4 \cdot 3 \cdot 2}$$

$$\int t^3 = \frac{t^4}{4}$$

Look for factorials

Offices hrs
today 4:30 - 5:30
See ICON for zoom room
(will be posted soon)

4.) Circle the general solution to the differential equation whose direction field is given below.

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A) $y = t + C$

B) $y = $ [illegible] $+ C$

C) $y = $ [illegible]

D) $y = Ce^t + t + 1$

E) $y = Ce^t$

F) $y = e^t + t + C$

G) $y = \ln(t) + C$

H) $y = C$

I) $y = \sin(t) + C$

J) $y = \cos(t) + C$

eq