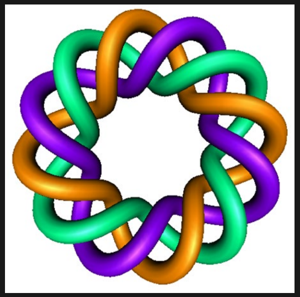
Existence and Uniqueness
+
Precalculus: Finding domain

example: $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$



By Dr. Isabel Darcy,
Dept of Mathematics and AMCS,
University of Iowa

2.4: Existence and Uniqueness

Thm 2.4.2: Suppose the functions

$$z = f(t, y) \text{ and } z = \frac{\partial f}{\partial y}(t, y)$$

are continuous on $(a, b) \times (c, d)$

and the point $(t_0, y_0) \in (a, b) \times (c, d)$,

then $\exists$ an interval $(t_0 - h, t_0 + h) \subset (a, b)$ such that $\exists!$ function $y = \phi(t)$ defined on $(t_0 - h, t_0 + h)$ that satisfies the following initial value problem:

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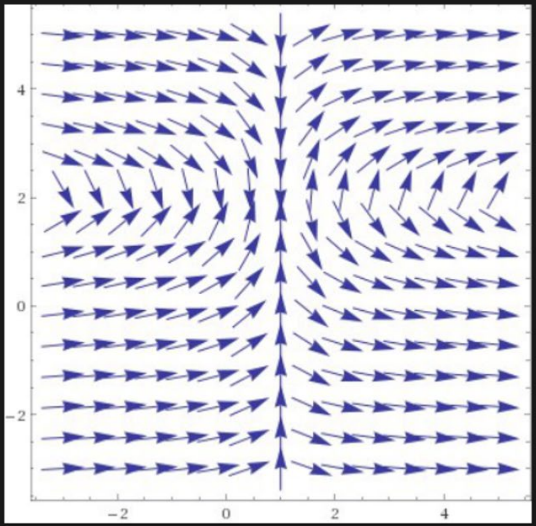

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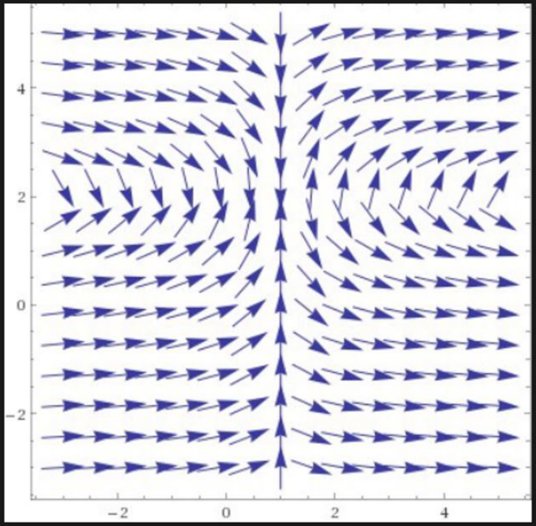

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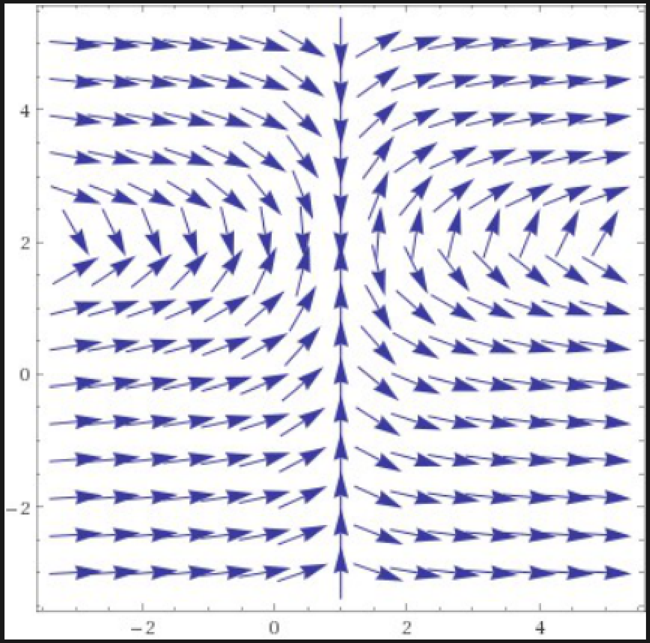
But what else can we say about this DE?

If $y_0 = 2$, $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$, $y(t_0) = 2$ has two solutions if $t_0 \neq 1$ (and if we allow vertical slope in domain. Note normally our convention will be to NOT allow vertical slope in domain of solution).



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If $t_0 = 1$, $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$, $y(1) = y_0$ has no solutions.

Solve via separation of variables: $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$

$$\int (2-y)dy = \int \frac{dt}{1-t}$$

$$2y - \frac{y^2}{2} = -ln|1-t| + C$$

$$y^2 - 4y - 2ln|1-t| + C = 0$$

$$y = \frac{4 \pm \sqrt{16+4(2ln|1-t|+C)}}{2} = 2 \pm \sqrt{4 + 2ln|1-t| + C}$$

$$y = 2 \pm \sqrt{2ln|1-t| + C}$$

IVP: $\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}, \quad y(0) = 1$

This IVP has a unique solution by thm 2.4.2.

General solution: $y = 2 \pm \sqrt{2ln|1-t| + C}$

Find C given $y(0) = 1$: $1 = 2 \pm \sqrt{2ln|1-0| + C}$

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$1 = 2 \pm \sqrt{C}$

$-1 = \pm\sqrt{C}$

$-1 = -\sqrt{C}$. Thus $C = 1$

IVP solution: $y = 2 - \sqrt{2ln|1-t| + 1}$

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Find domain:

NOTE: the convention in this class is to choose largest possible connected domain where tangent line to solution is never vertical.

Can't take square root of negative number:

Domain of ln is positive reals:

Cannot divide by 0:

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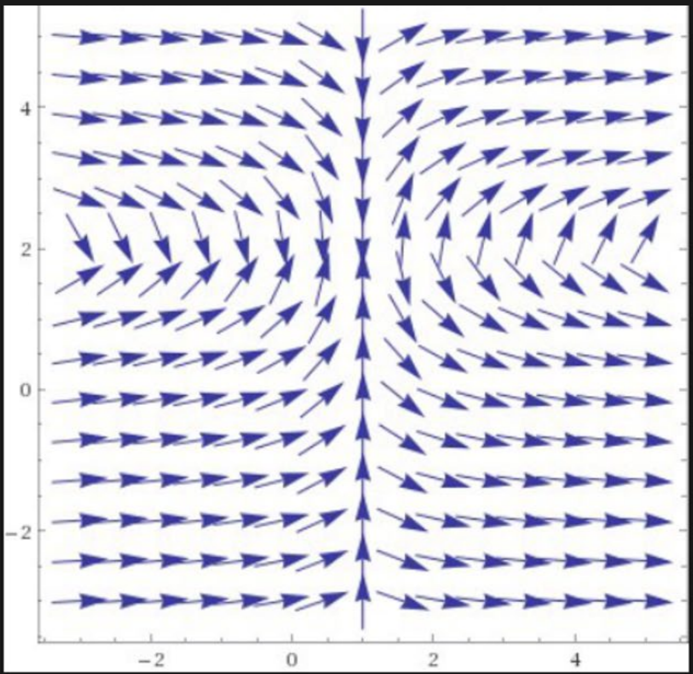

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$$2ln|1-t|+1 > 0$$

Domain of ln is positive reals: $|1-t| > 0 \Rightarrow t \neq 1$

Cannot divide by 0: Not Applicable to this problem.

$$2ln|1-t|+1>0$$

$2ln|1-t|>-1$ implies

$ln|1-t|>-\frac{1}{2}$

$|1-t|>e^{-\frac{1}{2}}$ since e^{t} is an increasing function.

$1-t<-e^{-\frac{1}{2}}$ or $1-t>e^{-\frac{1}{2}}$

$-t<-e^{-\frac{1}{2}}-1$ or $-t>e^{-\frac{1}{2}}-1$

$t>e^{-\frac{1}{2}}+1$ or $t<-e^{-\frac{1}{2}}+1$

Since IVP is $y(0)=1,\ t=0$ must be in domain:

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Thus $t<-e^{-\frac{1}{2}}+1$

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Find domain:

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$$2ln|1-t|+1 > 0$$

Domain of ln is positive reals: $|1-t| > 0 \Rightarrow t \neq 1$

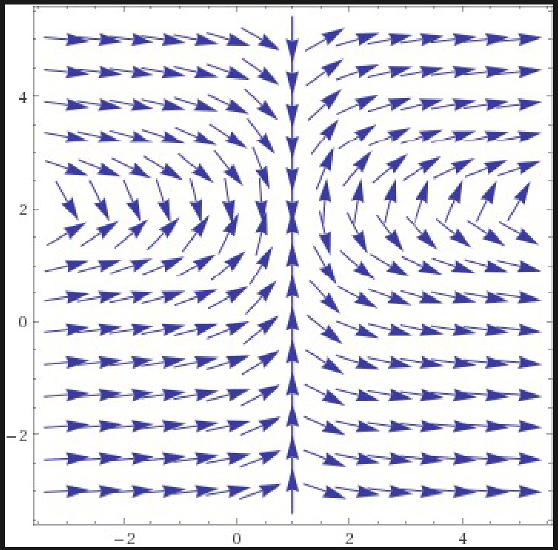
Cannot divide by 0: Not Applicable to this problem.

$t \neq 1$ and $t < -e^{-\frac{1}{2}} + 1$ implies $t < -e^{-\frac{1}{2}} + 1$

Thus domain is $t < -e^{-\frac{1}{2}} + 1$

IVP solution: $y = 2 - \sqrt{2ln|1-t|+1}$

with domain $(-\infty, -e^{-\frac{1}{2}} + 1)$



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