Solving Bernoulli's equation

$$y' + p(t)y = g(t)y^n$$

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2.4 Solve Bernoulli's equation,

$$y' + p(t)y = g(t)y^n$$

- If $n = 0$, then $y' + p(t)y = g(t)$ is linear.
- If $n = 1$, then $y' + p(t)y = g(t)y$ is separable.

When $n \neq 0, 1$, solve Bernoulli's by changing it to a linear equation.

How do you change a problem that you do not know how to solve into a problem that you do know how to solve?

The algorithm is NOT as important as the **idea** behind the algorithm.

Goal: Create linear equation, $w' + P(t)w = G(t)$

2.4 Solve Bernoulli's equation when $n \neq 0, 1$ by changing it to a linear equation

$$y' + p(t)y = g(t)y^n$$

Goal: Create linear equation, $w' + P(t)w = G(t)$

2.4 Solve Bernoulli's equation,

$$y' + p(t)y = g(t)y^n,$$

when $n \neq 0, 1$ by changing it

$$y^{-n}y' + p(t)y^{1-n} = g(t)$$

when $n \neq 0, 1$ by changing it to a linear equation by substituting $v = y^{1-n}$

and noting that $v' = (1-n)y^{-n}y'$

Goal: Create linear equation, $w' + P(t)w = G(t)$

Solve $ty' + 2t^{-2}y = 2t^{-2}y^5$

Goal: Create linear equation, $w' + P(t)w = G(t)$

Solve $ty' + 2t^{-2}y = 2t^{-2}y^5$

$$ty^{-5}y' + 2t^{-2}y^{-4} = 2t^{-2}$$

Let $v = y^{-4}$. Thus $v' = -4y^{-5}y'$

$$-4ty^{-5}y' - 8t^{-2}y^{-4} = -8t^{-2}$$

$$tv' - 8t^{-2}v = -8t^{-2}$$

We now have a linear DE that we can solve for v

$$tv' - 8t^{-2}v = -8t^{-2}$$

Solve linear DE for v:

Make coefficient of $v' = 1$

$$v' - 8t^{-3}v = -8t^{-3}$$

Find integrating factor:

An antiderivative of $-8t^{-3}$ is $4t^{-2}$

Thus integrating factor is $u(t) = e^{\int -8t^{-3}dt} = e^{4t^{-2}}$.

Multiply equation by $e^{4t^{-2}}$

$$e^{4t^{-2}}v' - 8t^{-3}e^{4t^{-2}}v = -8t^{-3}e^{4t^{-2}}$$

$$e^{4t^{-2}}v' - 8t^{-3}e^{4t^{-2}}v = -8t^{-3}e^{4t^{-2}}$$

$$(e^{4t^{-2}}v)' = -8t^{-3}e^{4t^{-2}} \text{ by PRODUCT rule.}$$

$$\int (e^{4t^{-2}}v)'dt = -8\int t^{-3}e^{4t^{-2}}dt$$

$$e^{4t^{-2}}v = -8\int t^{-3}e^{4t^{-2}}dt.$$

Integrate RHS using substitution:

Let $u = 4t^{-2}$. Then $du = -8t^{-3}dt$

$$e^{4t^{-2}}v = \int e^u du = e^u + C$$

$$e^{4t^{-2}}v = e^{4t^{-2}} + C$$

$$v = 1 + Ce^{-4t^{-2}}$$

$$v = 1 + Ce^{-4t^{-2}}$$

Solve $ty' + 2t^{-2}y = 2t^{-2}y^5$

Recall $v = y^{-4}$:

$y^{-4} = 1 + Ce^{-4t^{-2}}$ implies $y = \pm(1 + Ce^{-4t^{-2}})^{-\frac{1}{4}}$

General solution: $y = \pm(1 + Ce^{-4t^{-2}})^{-\frac{1}{4}}$

OR $y = 0$