To solve 1rst order linear DE: $1y' + p(t)y = g(t)$

Create product rule by using integrating factor: $u(t) = e^{\int p(t)dt}$

Ex 1: $ty' + 9y = \frac{e^{2t}}{t^5} \qquad \Rightarrow \qquad 1y' + \frac{9}{t}y = \frac{e^{2t}}{t^6}$

Create product rule by using integrating factor:

$$u(t) = e^{\int p(t)dt} = e^{\int \frac{9}{t}dt} = e^{9ln(t)dt} = e^{ln(t)^9} = t^9$$

$$t^9(y' + \frac{9}{t}y) = t^9(\frac{e^{2t}}{t^6})$$

$$t^9y' + 9t^8y = t^3e^{2t}$$

$$(t^9y)' = t^3e^{2t} \qquad \Rightarrow \qquad \int(t^9y)' = \int t^3e^{2t}dt$$

$$t^9y = \int t^3e^{2t}dt = t^3e^{2t} - \int 3t^2\frac{e^{2t}}{2}dt$$

$$t^9y = t^3\frac{e^{2t}}{2} - \frac{3t^2e^{2t}}{4} + \frac{6te^{2t}}{8} - \frac{6e^{2t}}{16} + C$$

General solution: $y = \frac{e^{2t}}{2t^6} - \frac{3e^{2t}}{4t^7} + \frac{3e^{2t}}{4t^8} - \frac{3e^{2t}}{8t^9} + \frac{C}{t^9}$

Use integration by parts on RHS

$u = t^3 \qquad dv = e^{2t}$

$du = 3t^2 \qquad v = \frac{e^{2t}}{2}$

$d^2u = 6t \qquad \int v = \frac{e^{2t}}{4}$

$d^3u = 6 \qquad \int\int v = \frac{e^{2t}}{8}$

$d^3u = 0 \qquad \int\int\int v = \frac{e^{2t}}{16}$

Ex 2: See ex 1 in https://homepage.math.uiowa.edu/ idarcy/ COURSES/100/SPRING21/2_1notesPM.pdf

Ex3: $2y' - \frac{4}{t}y = \frac{6t^2}{(t-1)(t+2)}$

$1y' - \frac{2}{t}y = \frac{3t^2}{(t-1)(t+2)}$

Create product rule by using integrating factor:

$u(t) = e^{\int p(t)dt} = e^{\int \frac{-2}{t}dt} = e^{-2ln(t)dt} = e^{ln(t)^{-2}} = t^{-2}$

$t^{-2}(y' - \frac{2}{t}y) = t^{-2}(\frac{3t^2}{(t-1)(t+2)})$

$t^{-2}y' - \frac{2}{t^{-3}}y = \frac{3}{(t-1)(t+2)}$

$(t^{-2}y)' = \frac{3}{(t-1)(t+2)}$

$\int(t^{-2}y)' = \int \frac{3}{(t-1)(t+2)}dt$

$t^{-2}y = \int \frac{3}{(t-1)(t+2)}dt$

Integration by partial fractions:

$\frac{3}{(t-1)(t+2)} = \frac{A}{t-1} + \frac{B}{t+2}$

$3 = A(t+2) + B(t-1)$

Method 1 for finding A and B:

Let $t = 1$: $\quad 3 = 3A$ and thus $A = 1$

Let $t = -2$: $\quad 3 = -3B$ and thus $B = -1$

Method 2 for finding A and B:

$0t + 3 = (A + B)t + 2A - B$

Comparing like terms:

$A + B = 0$ and
$2A - B = 3$

$3A = 3$ and thus $A = 1$

$B = -A = -1$

$t^{-2}y = \int \frac{3}{(t-1)(t+2)} dt = \int (\frac{1}{t-1} - \frac{1}{t+2}) dt = ln|t-1| - ln|t+2| + C$

General solutions: $y = t^2 ln|t-1| - t^2 ln|t+2| + Ct^2$

Solve via separation of variables (since not linear):

$$y' = \frac{1}{2-2t-y+ty}$$

$\frac{dy}{dt} = \frac{1}{2(1-t)-y(1-t)}$

$\frac{dy}{dt} = \frac{1}{(1-t)(2-y)}$

$\int (2-y)dy = \int \frac{dt}{1-t}$

$2y - \frac{y^2}{2} = -ln|1-t| + C$

$y^2 - 4y - 2ln|1-t| + C = 0$

$y = \frac{4 \pm \sqrt{16+4(2ln|1-t|+C)}}{2} = 2 \pm \sqrt{4 + 2ln|1-t| + C}$

$$\text{General solution: } y = 2 \pm \sqrt{2ln|1-t| + C}$$

IVP: $y(0) = 1$

$1 = 2 \pm \sqrt{2ln|1-0| + C}$

$1 - 2 = \pm\sqrt{2ln|1| + C}$

$-1 = \pm\sqrt{C}$

Choose negative sign since both sides must be negative.

$-1 = -\sqrt{C}$. Thus $C = 1$

$$\text{IVP solution: } y = 2 - \sqrt{2ln|1-t| + 1}$$

Find domain:

$2ln|1-t| + 1 \geq 0$ and $t \neq 1$ and $y \neq 2$

NOTE: the convention in this class to to choose largest possible connected domain where tangent line to solution is never vertical.

$2ln|1-t| \geq -1$ and $t \neq 1$ and $y \neq 2$ implies

$ln|1-t| > -\frac{1}{2}$

$|1-t| > e^{-\frac{1}{2}}$ since e^x is an increasing function.

$1-t < -e^{-\frac{1}{2}}$ or $1-t > e^{-\frac{1}{2}}$

$-t < -e^{-\frac{1}{2}} - 1$ or $-t > e^{-\frac{1}{2}} - 1$

Domain: $\begin{cases} t > e^{-\frac{1}{2}} + 1 & \text{if } t_0 > 1 \\ t < -e^{-\frac{1}{2}} + 1 & \text{if } t_0 < 1. \end{cases}$

Since $t_0 = 0$, Domain is $t < -e^{-\frac{1}{2}} + 1$