

$$g(t) = \begin{cases} 0 & t < 4 \\ 2 & 4 \leq t < 10 \\ t & t \geq 10 \end{cases}$$

Hence $g(t) = 2u_4(t) + (t - 2)u_{10}(t)$

Linearity

Solve $3y'' + y' + y = 2u_4(t) + (t - 2)u_{10}(t)$,
 $y(0) = 0, y'(0) = 0.$

$$3\mathcal{L}(y'') + \mathcal{L}(y') + \mathcal{L}(y) = \mathcal{L}(2u_4(t)) + \mathcal{L}((t - 2)u_{10}(t))$$

Formula 13 Thm: $\mathcal{L}(u_c(t)f(t - c)) = e^{-cs}\mathcal{L}(f(t)) = e^{-cs}F(s)$

Form 13 translated Thus $\mathcal{L}(u_c(t)f(t)) = e^{-cs}\mathcal{L}(f(t))$

Formula 18

$$3[s^2\mathcal{L}(y) - sy(0) - y'(0)] + s\mathcal{L}(y) - y(0) + \mathcal{L}(y) = e^{-4s}\mathcal{L}(2) + e^{-10s}\mathcal{L}((t + 8))$$

$$3[s^2\mathcal{L}(y)] + s\mathcal{L}(y) + \mathcal{L}(y) = 2e^{-4s}\mathcal{L}(1) + e^{-10s}\mathcal{L}(t) + 8e^{-10s}\mathcal{L}(1)$$

$$\mathcal{L}(y)[3s^2 + s + 1] = e^{-4s}\frac{2}{s} + e^{-10s}\frac{1}{s^2} + e^{-10s}\frac{8}{s}$$

solve for y

$$\mathcal{L}(y) = e^{-4s}\frac{2}{s[3s^2+s+1]} + e^{-10s}\frac{1}{s^2[3s^2+s+1]} + 8e^{-10s}\frac{1}{s[3s^2+s+1]}$$

$$y = 2\mathcal{L}^{-1}\left(e^{-4s}\frac{1}{s[3s^2+s+1]}\right) + \mathcal{L}^{-1}\left(e^{-10s}\frac{1}{s^2[3s^2+s+1]}\right) + 8\mathcal{L}^{-1}\left(e^{-10s}\frac{1}{s[3s^2+s+1]}\right)$$

(c=4) e^{-4s} F(s) *(c=10) e^{-10s} F(s)*

Characteristic polynomial

CHECK THIS!!!

$$g(t) = \begin{cases} 0 & t < 4 \\ 2 & 4 \leq t < 10 \\ t & t \geq 10 \end{cases}$$

Hence $g(t) = 2u_4(t) + (t-2)u_{10}(t)$

Solve $3y'' + y' + y = 2u_4(t) + (t-2)u_{10}(t),$
 $y(0) = 0, y'(0) = 0.$

$$3\mathcal{L}(y'') + \mathcal{L}(y') + \mathcal{L}(y) = \mathcal{L}(2u_4(t)) + \mathcal{L}((t-2)u_{10}(t))$$

Thm: $\mathcal{L}(u_c(t)f(t-c)) = e^{-cs}\mathcal{L}(f(t)).$

Thus $\mathcal{L}(u_c(t)f(t)) = e^{-cs}\mathcal{L}(f(t+c))$

$$3[s^2\mathcal{L}(y) - sy(0) - y'(0)] + s\mathcal{L}(y) - y(0) + \mathcal{L}(y) = e^{-4s}\mathcal{L}(2) + e^{-10s}\mathcal{L}((t+8))$$

$$3[s^2\mathcal{L}(y)] + s\mathcal{L}(y) + \mathcal{L}(y) = 2e^{-4s}\mathcal{L}(1) + e^{-10s}\mathcal{L}(t) + 8e^{-10s}\mathcal{L}(1)$$

$$\mathcal{L}(y)[3s^2 + s + 1] = e^{-4s}\frac{2}{s} + e^{-10s}\frac{1}{s^2} + e^{-10s}\frac{8}{s}$$

$$\mathcal{L}(y) = e^{-4s}\frac{2}{s[3s^2+s+1]} + e^{-10s}\frac{1}{s^2[3s^2+s+1]} + 8e^{-10s}\frac{1}{s[3s^2+s+1]}$$

$$y = 2\mathcal{L}^{-1}\left(e^{-4s}\frac{1}{s[3s^2+s+1]}\right) + \mathcal{L}^{-1}\left(e^{-10s}\frac{1}{s^2[3s^2+s+1]}\right) + 8\mathcal{L}^{-1}\left(e^{-10s}\frac{1}{s[3s^2+s+1]}\right)$$

characteristic polynomial

★ CHECK THIS!!!

Formula 13: $\mathcal{L}^{-1}(e^{-cs}F(s)) = u_c(t)f(t-c)$

$$y = 2u_4(t)f(t-4) + u_{10}h(t-10) + 8u_{10}f(t-10)$$

$$\text{where } f(t) = \mathcal{L}^{-1}\left(\frac{1}{s[3s^2+s+1]}\right) \text{ and } h(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2[3s^2+s+1]}\right)$$

F(s) H(s)

$$\frac{1}{s[3s^2+s+1]} = \frac{A}{s} + \frac{Bs+C}{3s^2+s+2}$$

Factors so
use partial
fractions

Does not fully
factor

$$1 = A(3s^2 + s + 1) + (Bs + C)s$$

$$0s^2 + 0s + 1 = (3A + B)s^2 + (A + C)s + A$$

$$0 = 3A + B, 0 = A + C, 1 = A$$

$$\text{Hence } A = 1, B = -3A = -3, C = -A = -1$$

$b^2 - 4ac = 1^2 - 4(3)(2) < 0$
Will eventually
need to
complete
the square

$$f(t) = \mathcal{L}^{-1}\left(\frac{1}{s[3s^2+s+1]}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{1}{s} + \frac{-3s-1}{3s^2+s+1}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{1}{s} + \frac{-3s-1}{3s^2+s+1}\right)$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-3s-1}{3[s^2+\frac{1}{3}s+\frac{1}{3}]}\right)$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-3s-1}{3[(s^2+\frac{1}{3}s+\frac{1}{3})-\frac{1}{36}+\frac{1}{3}]}\right)$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-3s-1}{3[(s+\frac{1}{6})^2-\frac{1}{36}+\frac{1}{3}]}\right)$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-3(s+\frac{1}{6})}{3[(s+\frac{1}{6})^2+\frac{11}{36}]}\right)$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-(s+\frac{1}{6}-\frac{1}{6}+\frac{1}{3})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right)$$

OK

$$= 1 + \mathcal{L}^{-1}\left(\frac{-(s+\frac{1}{6}+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right)$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} + \frac{-\frac{1}{6}}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right)$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} + \frac{-\frac{1}{6} \frac{6}{\sqrt{11}} \frac{\sqrt{11}}{6}}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right)$$

$$= 1 + \mathcal{L}^{-1}\left(\frac{-(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} + \frac{-\frac{1}{\sqrt{11}} \frac{\sqrt{11}}{6}}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right)$$

Thm: $\mathcal{L}^{-1}(F(s-c)) = e^{ct} \mathcal{L}^{-1}(F(s))$

$$= 1 + e^{-\frac{1}{6}t} \mathcal{L}^{-1}\left(\frac{-s}{[s^2+\frac{11}{36}]}\right) - \frac{1}{\sqrt{11}} e^{-\frac{1}{6}t} \mathcal{L}^{-1}\left(\frac{\frac{\sqrt{11}}{6}}{[s^2+\frac{11}{36}]}\right)$$

$$= 1 - e^{-\frac{1}{6}t} \cos \frac{\sqrt{11}}{6} t - \frac{1}{\sqrt{11}} e^{-\frac{1}{6}t} \sin \frac{\sqrt{11}}{6} t = f(t)$$

optional

$$h(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2[3s^2+s+1]}\right)$$

factors so partial fractions

$$\frac{1}{s^2[3s^2+s+1]} = \frac{As+D}{s^2} + \frac{Bs+C}{3s^2+s+1}$$

$$1 = (As+D)(3s^2+s+1) + (Bs+C)s^2$$

$$0s^3 + 0s^2 + 0s + 1 = (3A+B)s^3 + (A+3D+C)s^2 + (A+D)s + D$$

$$0 = 3A+B, 0 = A+3D+C, 0 = A+D, 1 = D.$$

Hence $D = 1, A = -D = -1, C = -A - 3D = 1 - 3 = -2, B = -3A = 3.$

Does not fully factor: $1^2 - 4(3)(1) < 0$ so will eventually need to complete the square

$$\frac{1}{s^2[3s^2+s+1]} = \frac{-s+1}{s^2} + \frac{3s-2}{3s^2+s+1} = \frac{-s}{s^2} + \frac{1}{s^2} + \frac{3(s-\frac{2}{3})}{3[(s+\frac{1}{6})^2+\frac{11}{36}]}$$

$$= \frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6}-\frac{1}{6}-\frac{2}{3})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} = \frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6}-\frac{5}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}$$

$$= \frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} - \frac{(\frac{5}{6})(\frac{6}{\sqrt{11}}\frac{\sqrt{11}}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}$$

$$= \frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} - \frac{(\frac{5}{6})(\frac{6}{\sqrt{11}}\frac{\sqrt{11}}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}$$

$$= \frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} - \frac{\frac{5}{\sqrt{11}}(\frac{\sqrt{11}}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}$$

$$h(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2[3s^2+s+1]}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{-1}{s} + \frac{1}{s^2} + \frac{(s+\frac{1}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]} - \frac{\frac{5}{\sqrt{11}}(\frac{\sqrt{11}}{6})}{[(s+\frac{1}{6})^2+\frac{11}{36}]}\right)$$

$$= -1 + t + e^{-\frac{1}{6}t} \cos \frac{\sqrt{11}}{6}t - \frac{5}{\sqrt{11}} e^{-\frac{1}{6}t} \sin \frac{\sqrt{11}}{6}t$$

Hence the final answer is

$$y = 2u_4(t)f(t-4) + u_{10}h(t-10) + 8u_{10}f(t-10)$$

$$\begin{aligned} &= 2u_4(t)[1 - e^{-\frac{1}{6}(t-4)} \cos \frac{\sqrt{11}}{6}(t-4) - \frac{1}{\sqrt{11}} e^{-\frac{1}{6}(t-4)} \sin \frac{\sqrt{11}}{6}(t-4)] + \\ &u_{10}[-1 + t - 10 + e^{-\frac{1}{6}(t-10)} \cos \frac{\sqrt{11}}{6}(t-10) - \frac{5}{\sqrt{11}} e^{-\frac{1}{6}(t-10)} \sin \frac{\sqrt{11}}{6}(t-10)] \\ &+ 8u_{10}[1 - e^{-\frac{1}{6}(t-10)} \cos \frac{\sqrt{11}}{6}(t-10) - \frac{1}{\sqrt{11}} e^{-\frac{1}{6}(t-10)} \sin \frac{\sqrt{11}}{6}(t-10)] \end{aligned}$$

TABLE 6.2.1 Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	Notes
1. 1	$\frac{1}{s}, \quad s > 0$	Sec. 6.1; Ex. 4
2. e^{at}	$\frac{1}{s-a}, \quad s > a$	Sec. 6.1; Ex. 5
3. $t^n, \quad n = \text{positive integer}$	$\frac{n!}{s^{n+1}}, \quad s > 0$	Sec. 6.1; Prob. 31
4. $t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$	Sec. 6.1; Prob. 31
5. $\sin at$	$\frac{a}{s^2 + a^2}, \quad s > 0$	Sec. 6.1; Ex. 7
6. $\cos at$	$\frac{s}{s^2 + a^2}, \quad s > 0$	Sec. 6.1; Prob. 6
7. $\sinh at$	$\frac{a}{s^2 - a^2}, \quad s > a $	Sec. 6.1; Prob. 8
8. $\cosh at$	$\frac{s}{s^2 - a^2}, \quad s > a $	Sec. 6.1; Prob. 7
9. $e^{at} \sin bt$	$\frac{b}{(s-a)^2 + b^2}, \quad s > a$	Sec. 6.1; Prob. 13
10. $e^{at} \cos bt$	$\frac{s-a}{(s-a)^2 + b^2}, \quad s > a$	Sec. 6.1; Prob. 14
11. $t^n e^{at}, \quad n = \text{positive integer}$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$	Sec. 6.1; Prob. 18
12. $u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$	Sec. 6.3
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$	Sec. 6.3
14. $e^{ct}f(t)$	$F(s-c)$	Sec. 6.3
15. $f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right), \quad c > 0$	Sec. 6.3; Prob. 25
16. $\int_0^t f(t-\tau)g(\tau) d\tau$	$F(s)G(s)$	Sec. 6.6
17. $\delta(t-c)$	e^{-cs}	Sec. 6.5
18. $f^{(n)}(t)$	$s^n F(s) - s^{n-1}f(0) - \dots - f^{(n-1)}(0)$	Sec. 6.2; Cor. 6.2.2
19. $(-t)^n f(t)$	$F^{(n)}(s)$	Sec. 6.2; Prob. 29

Handwritten notes in blue ink:

- Under row 17: n primes
- Under row 18: s^n s's, ie s^n