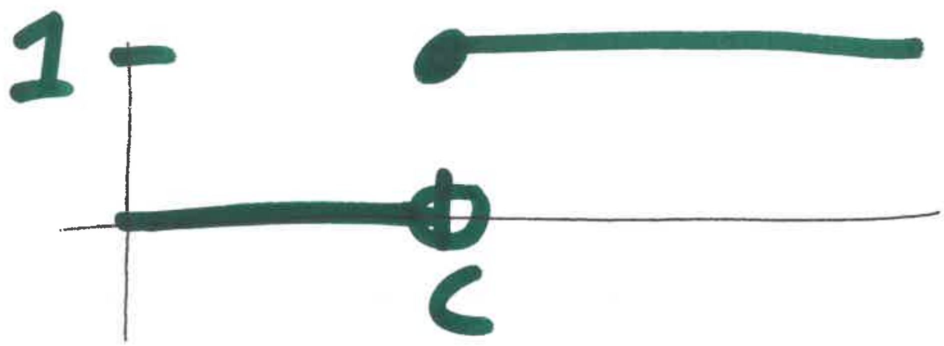
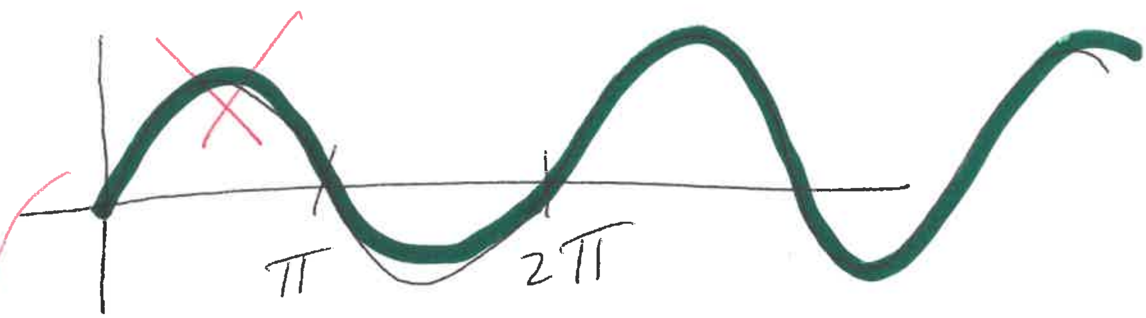
6.3: Step functions.

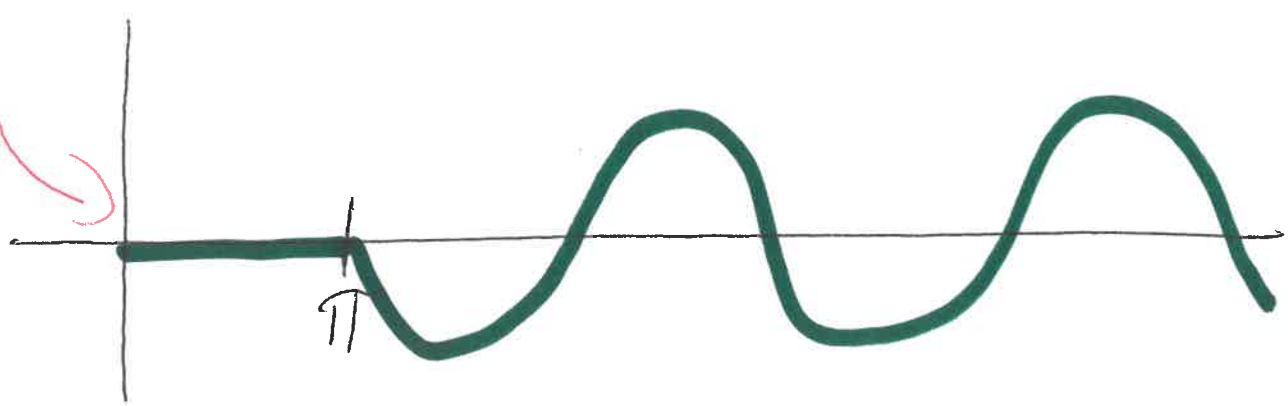
Graph $u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$



Graph $g(t) = sin(t)$.



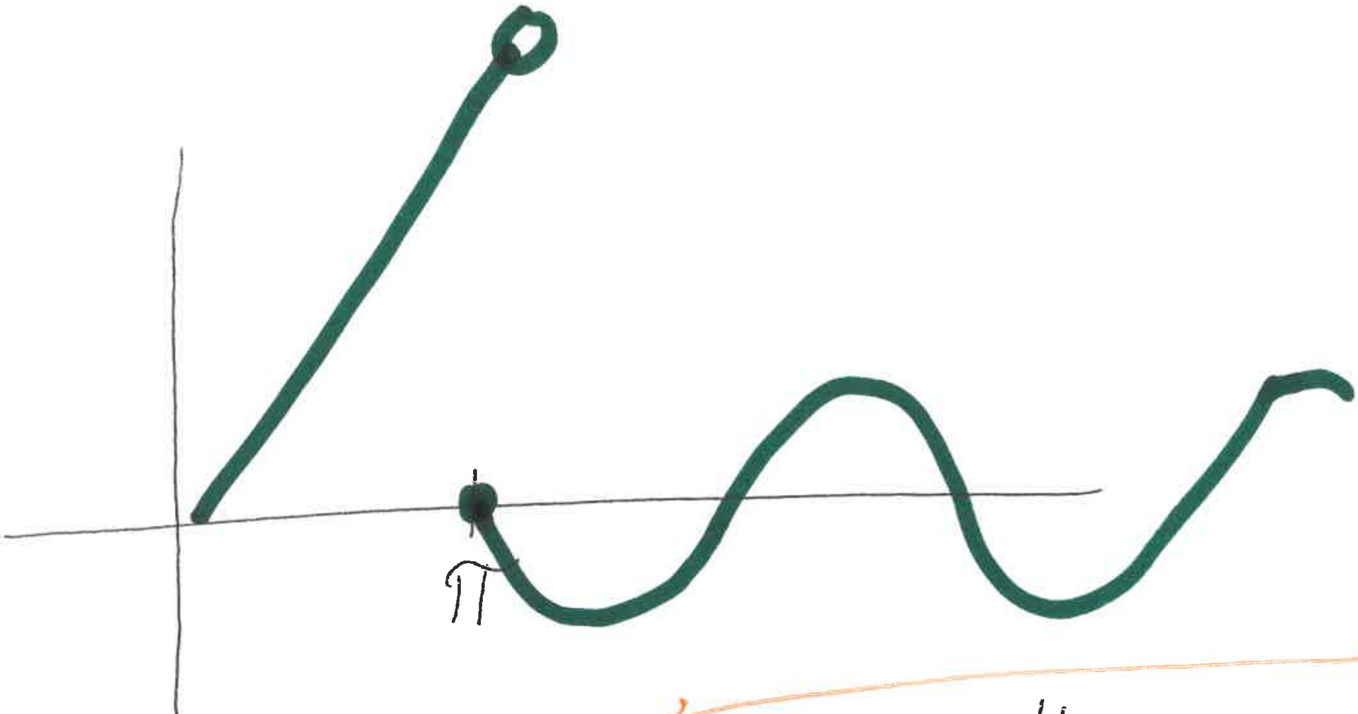
Graph $h(t) = u_\pi(t) sin(t)$. $= \begin{cases} 0 & t < \pi \\ \sin(t) & t \geq \pi \end{cases}$



Graph $f(t) = 2t + u_\pi(t)[sin(t) - 2t] = \begin{cases} 2t & t < \pi \\ \sin t & t \geq \pi \end{cases}$

$t < \pi$: $2t + 0$

$t \geq \pi$: $\cancel{2t} + (1)[\sin(t) - \cancel{2t}]$



$$\boxed{\text{If } t = 4 \quad u_4(t) = \begin{cases} 0 & t < 4 \\ 1 & t \geq 4 \end{cases}}$$

$$h(t) = \begin{cases} t & 0 \leq t < 4 \\ ln(t) & t \geq 4 \end{cases}$$

implies $h(t) = t + u_4(t)[\ln(t) - t]$

Partial check

$h(1) = t + 0[\ln(t) - t] = t$

$1 < 4$ $\qquad$ $t < 4$

$h(5) = \cancel{t} + 1[\ln(t) - \cancel{t}] = \ln(t)$

$t > 4$

Example: $\boxed{f(t) = \begin{cases} f_1, & \text{if } t < 4; \\ f_2, & \text{if } 4 \le t < 5; \\ f_3, & \text{if } 5 \le t < 10; \\ f_4, & \text{if } t \ge 10; \end{cases}}$

Hence

$$f(t) = f_1(t) + u_4(t)[f_2(t) - f_1(t)] + u_5(t)[f_3(t) - f_2(t)] \\ + u_{10}(t)[f_4(t) - f_3(t)]$$

Partial check:

If $t = 3$: $f(3) = f_1(3) + 0[f_2(3) - f_1(3)]$

$$+0[f_3(3) - f_2(3)] + 0[f_4(3) - f_3(3)] = f_1(3)$$

If $t = 9$: $f(9) = f_1(9) + 1[f_2(9) - f_1(9)]$

$$+1[f_3(9) - f_2(9)] + 0[f_4(9) - f_3(9)] = f_3(9)$$

Examples:

$f(t) = \begin{cases} 0 & 0 \le t < 2 \\ t^2 & t \ge 2 \end{cases}$ implies $f(t) =$

$g(t) = \begin{cases} t^2 & 0 \le t < 3 \\ 0 & t \ge 3 \end{cases}$ implies $g(t) =$

$$j(t) = \begin{cases} t & 0 \le t < 5 \\ 2 & 5 \le t \le 8 \\ e^t & t \ge 8 \end{cases} \quad \text{implies}$$

$$j(t) =$$

Formula 13: $\mathcal{L}(u_c(t) f(t-c)) = e^{-cs} F(s)$

Let $g(t) = f(t+c)$

$\Rightarrow g(t-c) = f(t - \cancel{c} + \cancel{c}) = f(t)$

$\mathcal{L}(u_c(t) f(t)) = \mathcal{L}(u_c(t) g(t-c))$

$\underset{\text{Formula 13}}{=} e^{-cs} G(s)$

$= e^{-cs} \mathcal{L}(g(t))$

$= e^{-cs} \mathcal{L}(f(t+c))$

$$\underbrace{u_c(t) f(t)}_{\text{Formula 13}'} \xrightarrow{\mathcal{L}} e^{-cs} \mathcal{L}(f(t+c))$$

for calculating inverse Laplace $\mathcal{L}^{-1}$

$\boxed{\text{Formula 13: } \mathcal{L}(u_c(t)f(t-c)) = e^{-cs}\mathcal{L}(f(t)). = e^{-cs}F(s)}$

Let $g(t) = f(t+c)$. Then $g(t-c) = f(t-c+c) = f(t)$. Thus

$$\mathcal{L}(u_c(t)f(t)) = \mathcal{L}(u_c(t)g(t-c)) = e^{-cs}\mathcal{L}(g(t))$$
$$= e^{-cs}\mathcal{L}(f(t+c)).$$

Formula 13'

or equivalently ← for calculating LaPlace

$$\mathcal{L}(u_c(t)f(t)) = e^{-cs}\mathcal{L}(f(t+c)).$$

$\mathcal{L}$

In other words, replacing $t-c$ with t is equivalent to replacing t with $t+c$

Find the LaPlace transform of the following:

a.) $\mathcal{L}(\underbrace{u_3(t)}_{c=3}\underbrace{(t^2-2t+1)}_{f(t)}) =$ ______

$$= e^{-3s}\mathcal{L}(\underbrace{(t+3)^2 - 2(t+3) + 1}_{f(t+c) = f(t+3)})$$

e^{-cs}

$$= e^{-3s}\mathcal{L}(t^2+6t+9-2t-6+1)$$

$c=3$

$$= e^{-3s}\mathcal{L}(t^2+4t+4\cdot 1)$$

$$e^{-3s}[\mathcal{L}(t^2)+4\mathcal{L}(t)+4\mathcal{L}(1)] = e^{-3s}\left[\frac{2}{s^3}+4\left(\frac{1}{s^2}\right)+\frac{4}{s}\right]$$

$c=4$

b.) $\mathcal{L}(u_4(t)(e^{-8t})) = e^{-4s}\mathcal{L}(f(t+4)) = e^{-4s}\left(\frac{e^{-32}}{s+8}\right)$

$= \frac{e^{32-4s}}{s+8}$

c.) $\mathcal{L}(u_2(t)(t^2e^{3t})) =$ ________________

Find the LaPlace transform of

$u_4(t)\,f(t)$
$= u_4(t)\,e^{-8t}$

$f(t) = e^{-8t}$

$\mathcal{L}(f(t+4)) = \mathcal{L}(e^{-8(t+4)})$

$= \mathcal{L}(e^{-8t-32}) = \mathcal{L}(e^{-8t}e^{-32})$

$e^{-32}\mathcal{L}(e^{-8t})$

$e^{-32}\left(\frac{1}{s-(-8)}\right)$

$= \frac{e^{-32}}{s+8}$

d.) $g(t) = \begin{cases} 0 & t < 3 \\ e^{t-3} & t \geq 3 \end{cases}$

e.) $f(t) = \begin{cases} 0 & t < 3 \\ 5 & 3 \leq t < 4 \\ t-5 & t \geq 4 \end{cases}$

Calculating $\mathcal{L}^{-1}$

Inverse La Place

Formula 13: $\mathcal{L}(u_c(t)f(t-c)) = e^{-cs}\mathcal{L}(f(t))$.

Let $F(s) = \mathcal{L}(f(t))$.

Then $\mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}(\mathcal{L}(f(t))) = f(t)$.

Thus $\mathcal{L}^{-1}(e^{-cs}F(s))$

$$= \mathcal{L}^{-1}(e^{-cs}\mathcal{L}(f(t))) = u_c(t)f(t-c)$$

1) determine $\mathcal{L}^{-1}(F) = f$

where $f(t) = \mathcal{L}^{-1}(F(s))$ 2) evaluat f at $t-c$

Find the inverse LaPlace transform of the following:

a.) $\mathcal{L}^{-1}(\underbrace{e^{-8s}}_{e^{-cs}}\underbrace{\tfrac{1}{s-3}}_{F(s)}) = u_8(t)\,f(t-8) = u_8(t)\,e^{3(t-8)}$

$c=+8$

where $f(t) = \mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}\left(\frac{1}{s-3}\right) = e^{3t} = f(t)$

b.) $\mathcal{L}(u_4(t)(e^{-8t})) =$ $e^{-4s}e^{-32}\mathcal{L}(e^{-8t}) = e^{-4s-32}\left(\frac{1}{s+8}\right)$

c.) $\mathcal{L}(u_2(t)(t^2e^{3t})) =$ ______

Find the LaPlace transform of

d.) $g(t) = \begin{cases} 0 & t < 3 \\ e^{t-3} & t \geq 3 \end{cases}$

e.) $f(t) = \begin{cases} 0 & t < 3 \\ 5 & 3 \leq t < 4 \\ t-5 & t \geq 4 \end{cases}$

$\mathcal{L}(\underbrace{u_4(t)e^{-8t}}) \quad c=4$

$= e^{-4s}\mathcal{L}(f(t+4))$

$= e^{-4s}\mathcal{L}(e^{-8(t+4)})$

$= e^{-4s}\mathcal{L}(e^{-8t-32})$

$= e^{-4s}\mathcal{L}(e^{-32}e^{-8t})$

Formula 13: $\mathcal{L}(u_c(t)f(t-c)) = e^{-cs}\mathcal{L}(f(t))$.

Let $F(s) = \mathcal{L}(f(t))$.

Then $\mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}(\mathcal{L}(f(t))) = f(t)$.

Thus $\mathcal{L}^{-1}(e^{-cs}F(s))$

$$= \mathcal{L}^{-1}(e^{-cs}\mathcal{L}(f(t))) = u_c(t)f(t-c)$$

where $f(t) = \mathcal{L}^{-1}(F(s))$

$u_c(t)\, f(t) =$

$u_4(t)\, e^{-8t}$

$c=4$

$f(t) = e^{-8t}$

$f(t+4) = e^{-8(t+4)}$

Inverse LaPlace

① Find f by calcuting $\mathcal{L}^{-1}(F)$

② Evaluate f at $t-c$

Find the inverse LaPlace transform of the following:

a.) $\mathcal{L}^{-1}(e^{-8s}\frac{1}{s-3}) =$ $u_8(t)\, f(t-8) = \boxed{u_8(t)\, e^{3(t-8)}}$

$c=8$

$\underbrace{e^{-cs}F(s)} \Rightarrow F(s) = \frac{1}{s-3} \Rightarrow f(t) = \mathcal{L}^{-1}(F) = \mathcal{L}^{-1}\left(\frac{1}{s-3}\right) = e^{3t}$

b.) $\mathcal{L}^{-1}(e^{-4s}\frac{1}{s^2-3}) =$ ______________________________

$c=1$

c.) $\mathcal{L}^{-1}(e^{-1s}\frac{5}{(s-3)^4}) = \frac{5}{6}e^{3(t-1)}(t-1)^3$

$\mathcal{L}^{-1}(e^{-cs}F(s)) = u_c(t)f(t-c) = u_1(t)f(t-1)$

where $f(t) = \mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}\left(\frac{5}{(s-3)^4}\right) =$

$= \frac{1}{3!}\mathcal{L}^{-1}\left(\frac{5\cdot 3!}{(s-3)^4}\right)$

d.) $\mathcal{L}^{-1}(\frac{e^{-s}}{4s}) =$ ______________________________

$= \frac{5}{6}\mathcal{L}^{-1}\left(\frac{3!}{(s-3)^4}\right) = \frac{5}{6}e^{3t}t^n$

e.) $\mathcal{L}^{-1}(e^{-s}) =$ ______________________________

f.) $\mathcal{L}^{-1}(e^{-s}\frac{1}{(s-3)^2+4}) =$ ______________________________ [redacted]

g.) $\mathcal{L}^{-1}(e^{-s}\frac{2s-5}{s^2+6s+13}) =$ ______________________________ [redacted] b.) $\mathcal{L}^{-1}(e^{-4s}\frac{1}{s^2-3}) =$ ______________________________

c.) $\mathcal{L}^{-1}(e^{-s}\frac{5}{(s-3)^4}) = u_1(t)f(t-1) = \boxed{u_1(t)\left[\frac{5}{6}(t-1)^3 e^{3(t-1)}\right]}$

$e^{-cs}F(s)$

$c = 1$

$\boxed{F(s) = \frac{5}{(s-3)^4}}$

$f(t) = \mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}\left(\frac{5}{(s-3)^4}\right)$

$n+1 = 4 \Rightarrow n = 3$

$= \frac{5}{3!}\mathcal{L}^{-1}\left(\frac{3!}{(s-3)^4}\right) = \frac{5}{6}t^3 e^{3t} = f(t)$

$a = 3$

d.) $\mathcal{L}^{-1}(\frac{e^{-s}}{4s}) =$ ______________________________

e.) $\mathcal{L}^{-1}(e^{-s}) =$ ______________________________

f.) $\mathcal{L}^{-1}(e^{-s}\frac{1}{(s-3)^2+4}) =$ ______________________________

g.) $\mathcal{L}^{-1}(e^{-s}\frac{2s-5}{s^2+6s+13}) =$ ______________________________