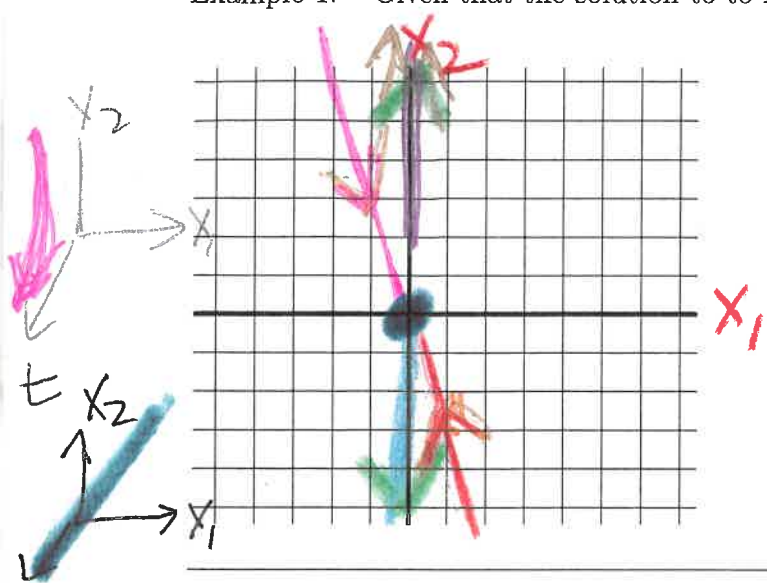


eigenvalues

7.5 Two real eigenvalues: Graph several trajectories for the following systems of equations:

Example 1: Given that the solution to $x' = \begin{bmatrix} -2 & 0 \\ 21 & 5 \end{bmatrix} x$ is $x = c_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t}$



The equilibrium solution for this system of equations is

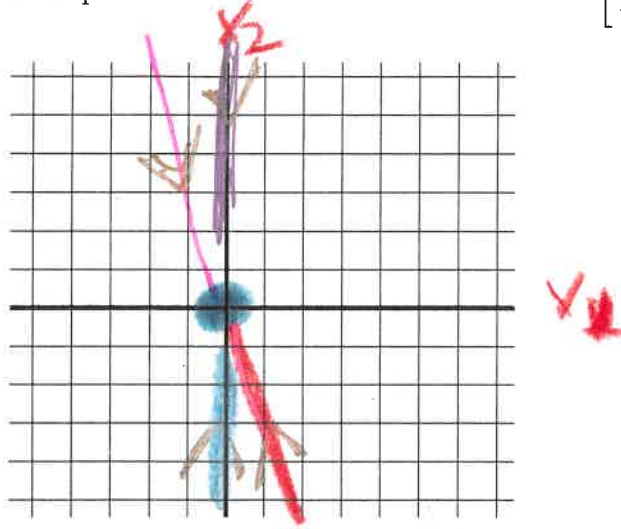
$$x_2 = \frac{3}{-1} x_1$$

This equilibrium solution is (choose one):
asymptotically stable or unstable

Classify this critical point's type:

$$x_2 = \frac{1}{0} x_1$$

Example 2: Given that the solution to $x' = \begin{bmatrix} -2 & 0 \\ -9 & -5 \end{bmatrix} x$ is $x = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-5t} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t}$



The equilibrium solution for this system of equations is

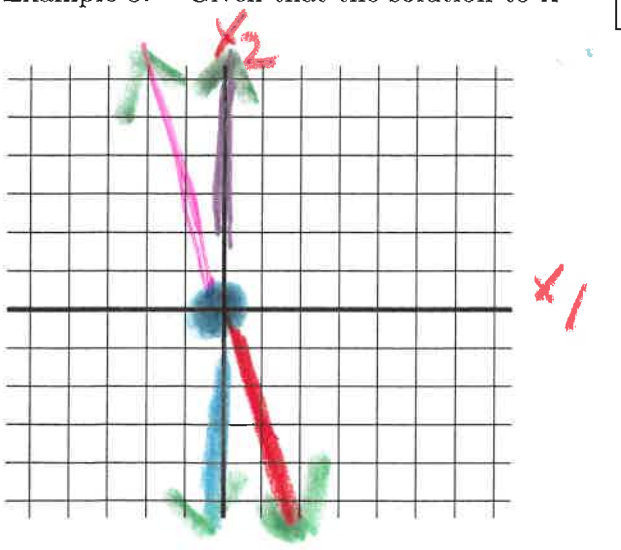
$$x_2 = \frac{3}{-1} x_1$$

This equilibrium solution is (choose one):
asymptotically stable or unstable

Classify this critical point's type:

$$x_2 = \frac{1}{0} x_1$$

Example 3: Given that the solution to $x' = \begin{bmatrix} 2 & 0 \\ 9 & 5 \end{bmatrix} x$ is $x = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{2t}$



The equilibrium solution for this system of equations is

$$x_2 = \frac{3}{-1} x_1$$

This equilibrium solution is (choose one):
asymptotically stable or unstable

Classify this critical point's type:

$$x_2 = \frac{1}{0} x_1$$

Example 1: Given that the solution to $x' = \begin{bmatrix} -2 & 0 \\ 21 & 5 \end{bmatrix} x$ is $x = c_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t}$

IVP: Plug in, solve for c_1 and c_2

$\Rightarrow c_1 = 1, c_2 = 0$

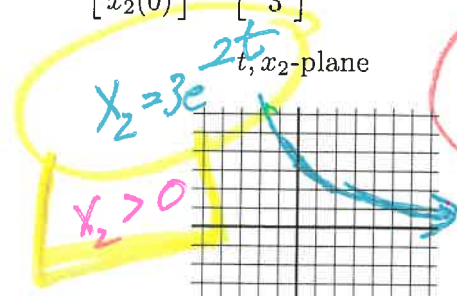
\Rightarrow soln: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t}$

Graph the solution to the IVP $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ in the

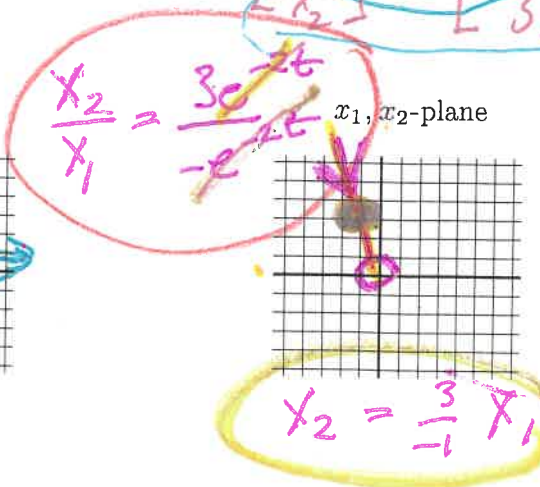
t, x_1 -plane



t, x_2 -plane

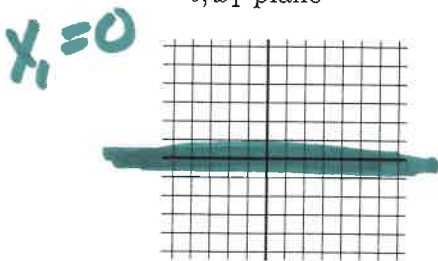


x_1, x_2 -plane

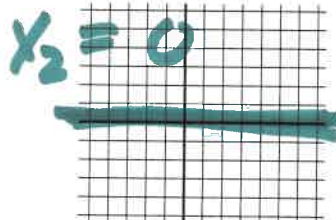


Graph the solution to the IVP $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in the

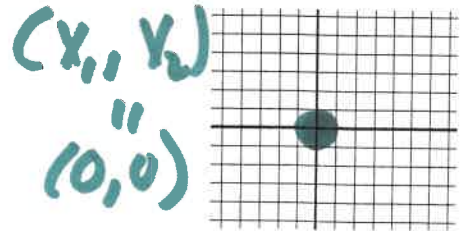
t, x_1 -plane



t, x_2 -plane



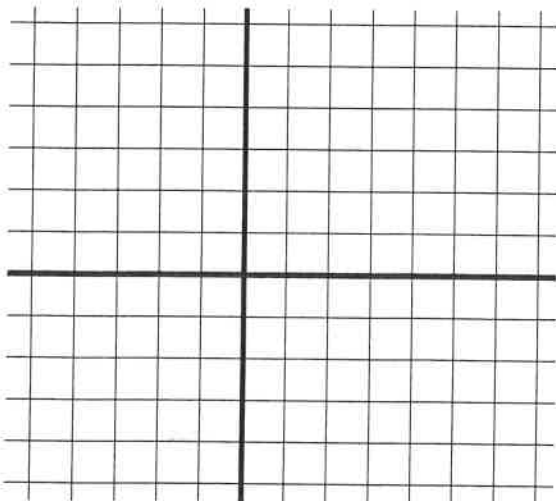
x_1, x_2 -plane



The equilibrium solution for this system of equations is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

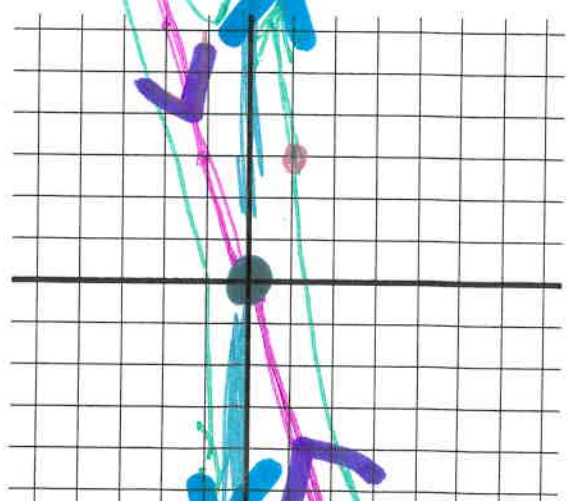
$\frac{dx_2}{dx_1} =$ _____

Plot several direction vectors where the slope is 0 and where slope is vertical.



$\begin{bmatrix} -1 \\ +3 \end{bmatrix} \rightarrow$ slope $\frac{3}{-1} = \text{neg exp}$

Graph several trajectories.



pos exp $\rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow$ slope ∞

Semi-generic ex: Given that the solution to $x' = \begin{bmatrix} -2 & 0 \\ 21 & 5 \end{bmatrix} x$ is $x = c_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{r_1 t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{r_2 t}$

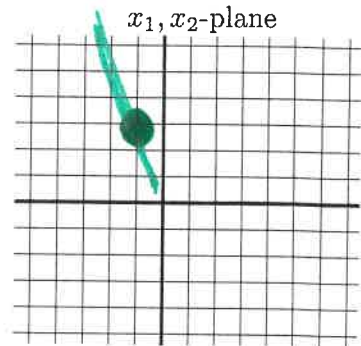
IVP: $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ implies $c_1 = 1, c_2 = 0$.

Thus IVP soln: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{r_1 t}$

Hence $x_1 = -e^{r_1 t} < 0$ and $x_2 = 3e^{r_1 t} > 0$

and $\frac{x_2}{x_1} = \frac{3e^{r_1 t}}{-e^{r_1 t}} = \frac{3}{-1}$. Thus $x_2 = \frac{3}{-1} x_1$.

<https://www.geogebra.org/3d> (t, -e^(-2t), 3*e^(-2t))



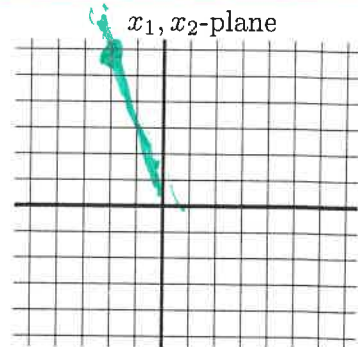
IVP: $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$ implies $c_1 = 2, c_2 = 0$.

Thus IVP soln: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{r_1 t}$

Hence $x_1 = -2e^{r_1 t} < 0$ and $x_2 = 6e^{r_1 t} > 0$

and $\frac{x_2}{x_1} = \frac{6e^{r_1 t}}{-2e^{r_1 t}} = \frac{3}{-1}$. Thus $x_2 = \frac{3}{-1} x_1$.

<https://www.geogebra.org/3d> (t, -2*e^(-2t), 6*e^(-2t))



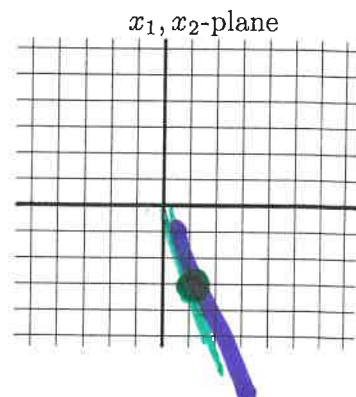
IVP: $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ implies $c_1 = -1, c_2 = 0$.

Thus IVP soln: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = - \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{r_1 t}$

Hence $x_1 = e^{r_1 t} > 0$ and $x_2 = -3e^{r_1 t} < 0$

and $\frac{x_2}{x_1} = \frac{-3e^{r_1 t}}{e^{r_1 t}} = \frac{-3}{1}$. Thus $x_2 = \frac{-3}{1} x_1$.

<https://www.geogebra.org/3d> (t, e^(-2t), -3*e^(-2t))



Semi-generic ex: Given that the solution to $x' = \begin{bmatrix} -2 & 0 \\ 21 & 5 \end{bmatrix} x$ is $x = c_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{r_1 t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{r_2 t}$

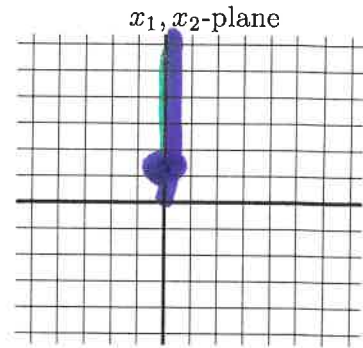
IVP: $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ implies $c_1 = 0, c_2 = 1$.

Thus IVP soln: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{r_2 t}$

Hence $x_1 = 0$ and $x_2 = e^{r_2 t} > 0$

and $\frac{x_2}{x_1} = \frac{1e^{r_2 t}}{0e^{r_2 t}} = \frac{1}{0}$. Thus $x_2 = \frac{1}{0}x_1$.

<https://www.geogebra.org/3d> (t, 0, e^(5t))



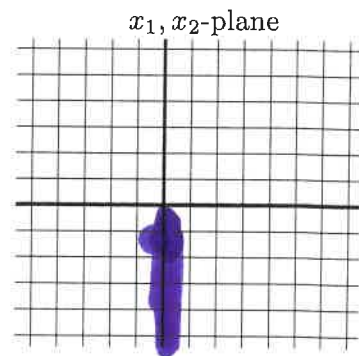
IVP: $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ implies $c_1 = 0, c_2 = -1$.

Thus IVP soln: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = - \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{r_2 t}$

Hence $x_1 = 0$ and $x_2 = -e^{r_2 t} < 0$

and $\frac{x_2}{x_1} = \frac{-1e^{r_2 t}}{0e^{r_2 t}} = \frac{-1}{0}$. Thus $x_2 = \frac{-1}{0}x_1$.

<https://www.geogebra.org/3d> (t, 0, -e^(5t))

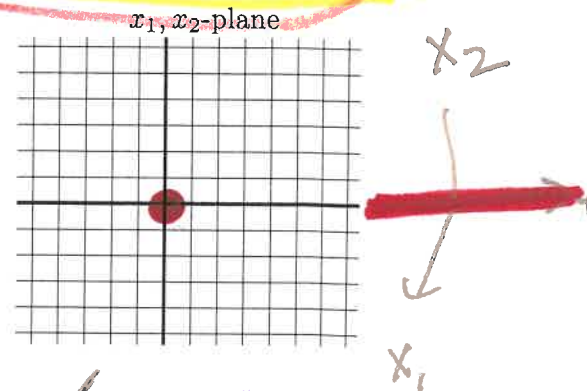


IVP: $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ implies $c_1 = c_2 = 0$.

Thus IVP soln: $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Hence $x_1 = 0$ and $x_2 = 0$

<https://www.geogebra.org/3d> (t, 0, 0)



$x_1(t) = 0$
 $x_2(t) = 0$ } constant soln
 ||
equilibrium soln

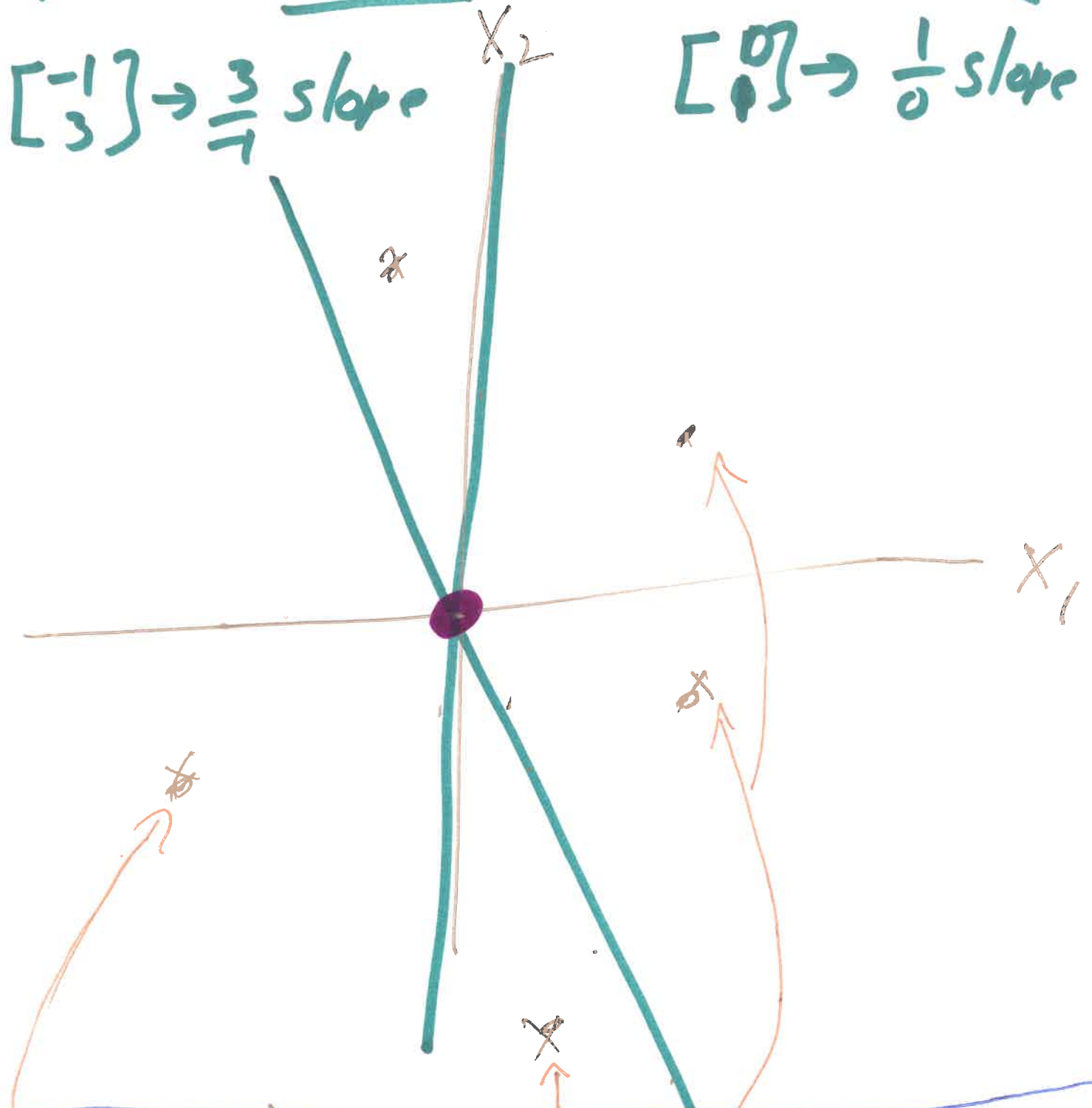
$\vec{x}' = A\vec{x} \Rightarrow \vec{x}(t) = 0$ is a soln $\Rightarrow \vec{x}(t) = 0$ is an equil soln

Two real

e. values

$\begin{bmatrix} -1 \\ 3 \end{bmatrix} \rightarrow \frac{3}{7}$ slope

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \frac{1}{0}$ slope



What if $c_1 \neq 0$ and $c_2 \neq 0$

Answer the following questions for $A = \begin{bmatrix} -2 & 0 \\ 21 & 5 \end{bmatrix}$:

The smaller eigenvalue of A is $r_1 = -2$. An eigenvector corresponding to r_1 is $\mathbf{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

The larger eigenvalue of A is $r_2 = 5$. An eigenvector corresponding to r_2 is $\mathbf{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

The general solution to $\mathbf{x}' = \begin{bmatrix} -2 & 0 \\ 21 & 5 \end{bmatrix} \mathbf{x}$ is $\mathbf{x} = c_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t}$.

$\begin{bmatrix} -1 \\ 3 \end{bmatrix} c_1 \neq 0$
 $c_2 \neq 0$

For large **positive** values of t which is larger: e^{-2t} or e^{5t} ?

$t \rightarrow +\infty$
 $e^{-2t} \rightarrow 0$
 $e^{5t} \rightarrow \infty$

For the following problems, consider the case when $c_1 \neq 0$ and $c_2 \neq 0$.

For large **positive** values of t , which term dominates: $c_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t}$ or $c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t}$?

e^{5t} dominates

Thus for large **positive** values of t , such trajectories (where $c_1 c_2 \neq 0$) when projected into the x_1, x_2 plane exhibit the following behavior (select all that apply):

5

* moves away from the origin.

* moves toward the origin.

* approaches the line $y = mx$ with slope $m = 1/0$.

* approaches a line $y = mx + b$ for $b \neq 0$ with slope $m =$ _____. Note this case corresponds to where both $\|c_1 \mathbf{v}\| e^{r_1 t}$ and $\|c_2 \mathbf{w}\| e^{r_2 t}$ are large, but one is significantly larger than the other.

positive exp $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
 $1/0$

$t = -1000$ $e^{2000} \gg e^{-5000}$

For large **negative** values of t which is larger: e^{-2t} or e^{5t} ?

$t \rightarrow -\infty$
 $e^{-2t} \rightarrow \infty$
 $e^{5t} \rightarrow 0$

For large **negative** values of t , which term dominates: $c_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t}$ or $c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t}$?

$c_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t}$ dominates

Thus for large **negative** values of t , such trajectories (where $c_1 c_2 \neq 0$) when projected into the x_1, x_2 plane exhibit the following behavior (select all that apply):

* moves away from the origin.

* moves toward the origin.

* approaches the line $y = mx$ with slope $m = 3/-1$.

* approaches a line $y = mx + b$ for $b \neq 0$ with slope $m =$ _____. Note this case corresponds to where both $\|c_1 \mathbf{v}\| e^{r_1 t}$ and $\|c_2 \mathbf{w}\| e^{r_2 t}$ are large, but one is significantly larger than the other.

backwards in time $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$

more backwards in time

One $+$ and one $-$ e. value

$$\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$



$\frac{3}{-1}$ slope

\ominus e. value

\Downarrow
negative exponential



$$\vec{x} \rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

as $t \rightarrow \infty$



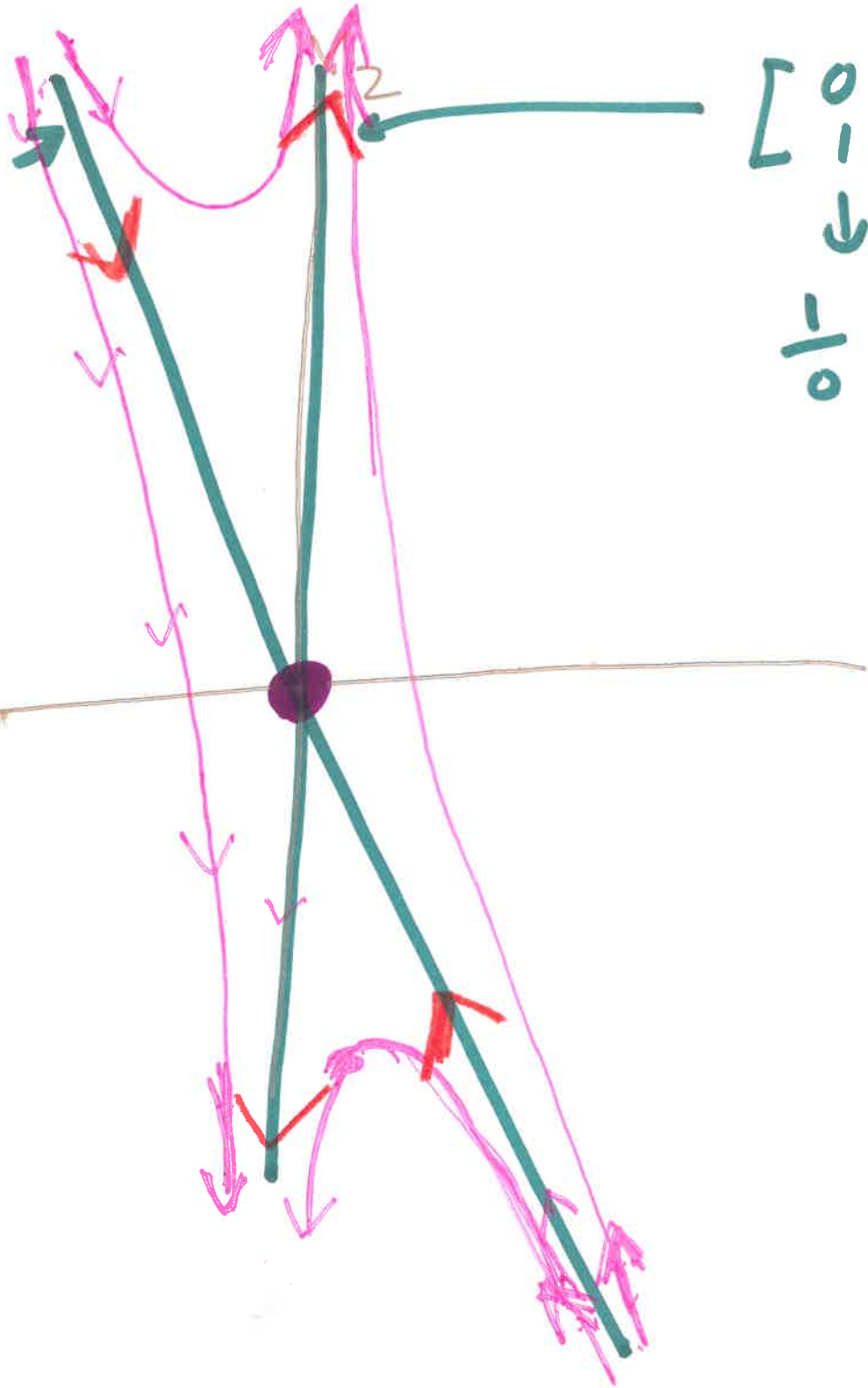
arrow points towards the origin

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



$\frac{1}{0}$ slope

\oplus e. value
value
 \downarrow
positive exponential



Answer the following questions for $A = \begin{bmatrix} 2 & 0 \\ 9 & 5 \end{bmatrix}$:

$c_1 \neq 0, c_2 \neq 0$

The smaller eigenvalue of A is $r_1 = 2$. An eigenvector corresponding to r_1 is $\mathbf{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

The larger eigenvalue of A is $r_2 = 5$. An eigenvector corresponding to r_2 is $\mathbf{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

The general solution to $\mathbf{x} = \begin{bmatrix} 2 & 0 \\ 9 & 5 \end{bmatrix} \mathbf{x}$ is $\mathbf{x} = c_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t}$

two positive e. values

$t = 1000 \quad e^{2000} \ll e^{5000}$

Two positive e. values

For large positive values of t which is larger: e^{2t} or e^{5t} ?

For the following problems, consider the case when $c_1 \neq 0$ and $c_2 \neq 0$ where the general solution is

$\mathbf{x} = c_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t}$ $t \rightarrow +\infty$

For large positive values of t , which term dominates: $c_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{2t}$ or $c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t}$

large very very large

Thus for large positive values of t , such trajectories (where $c_1 c_2 \neq 0$) when projected into the x_1, x_2 plane exhibit the following behavior (select all that apply):

- * moves away from the origin. $e^{pos exp}$
- * moves toward the origin.
- * approaches the line $y = mx$ with slope $m = \frac{1}{0}$
- * approaches a line $y = mx + b$ for $b \neq 0$ with slope $m = \frac{1}{0}$. Note this case corresponds to where both $\|c_1 \mathbf{v}\| e^{r_1 t}$ and $\|c_2 \mathbf{w}\| e^{r_2 t}$ are large, but one is significantly larger than the other.

$t = -1000 \quad e^{-2000} \gg e^{-5000}$

For large negative values of t which is larger: e^{2t} or e^{5t} ?

For large negative values of t , which term dominates: $c_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{2t}$ or $c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t}$

small tiny

Thus for large negative values of t , such trajectories (where $c_1 c_2 \neq 0$) when projected into the x_1, x_2 plane exhibit the following behavior (select all that apply):

- * moves away from the origin.
- * moves toward the origin.
- * approaches the line $y = mx$ with slope $m = \frac{3}{-1}$
- * approaches a line $y = mx + b$ for $b \neq 0$ with slope $m = \frac{3}{-1}$. Note this case corresponds to where both $\|c_1 \mathbf{v}\| e^{r_1 t}$ and $\|c_2 \mathbf{w}\| e^{r_2 t}$ are large, but one is significantly larger than the other.

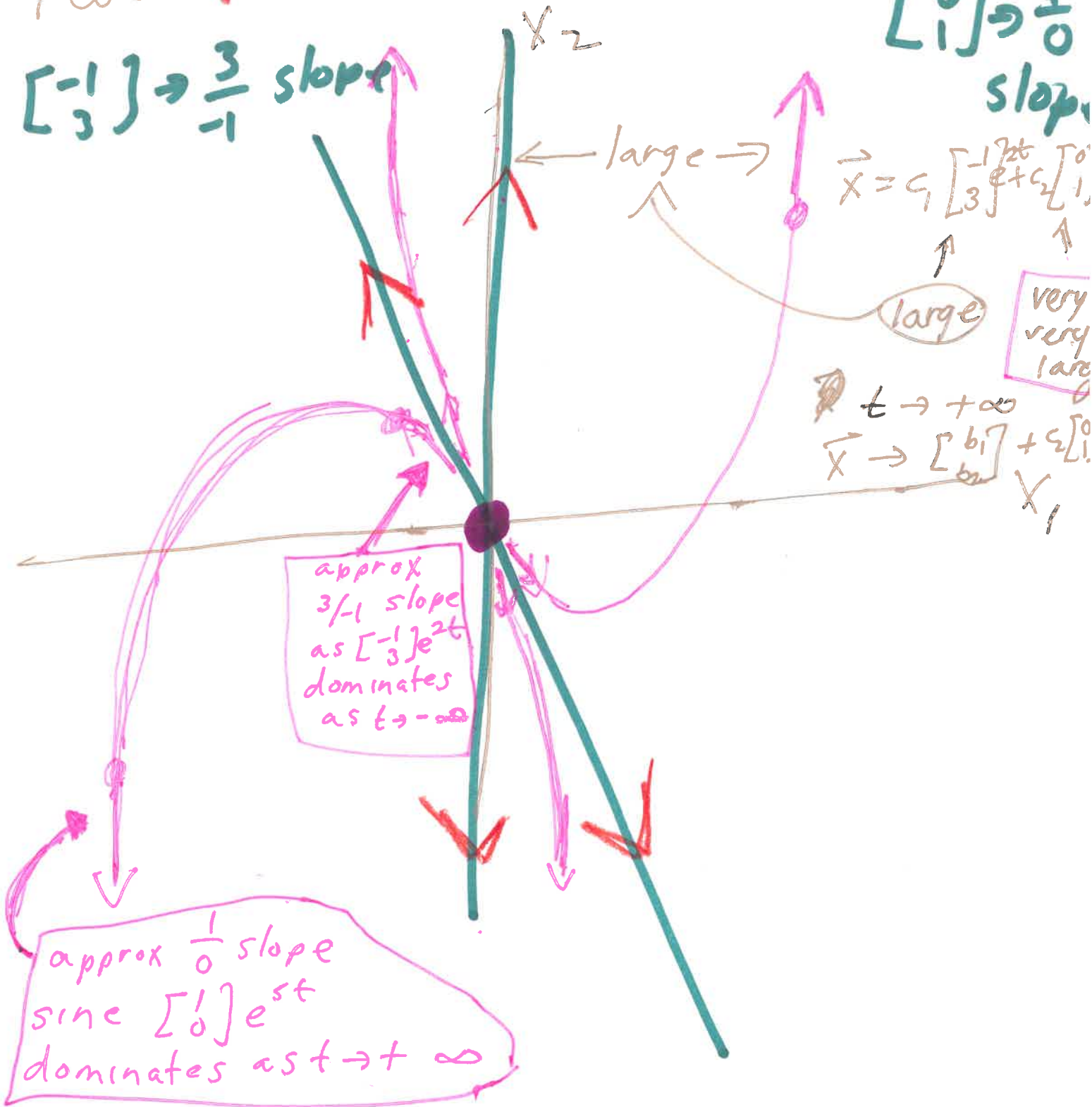
$t \rightarrow -\infty$
 $\mathbf{x} \rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

backwards in time for large $-t$, start near the origin

Two + e. values

$[-1/3] \rightarrow \frac{3}{-1}$ slope

$[0/1] \rightarrow \frac{0}{1}$ slope



positive exponential go away from origin

Answer the following questions for $A = \begin{bmatrix} -2 & 0 \\ -9 & -5 \end{bmatrix}$:

The smaller eigenvalue of A is $r_1 = -5$. An eigenvector corresponding to r_1 is $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

The larger eigenvalue of A is $r_2 = -2$. An eigenvector corresponding to r_2 is $\mathbf{w} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

The general solution to $\mathbf{x}' = \begin{bmatrix} -2 & 0 \\ -9 & -5 \end{bmatrix} \mathbf{x}$ is $\mathbf{x} = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-5t} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t}$

$t = 1000$ e^{-5000} \leftarrow e^{-2000} \leftarrow larger $t \rightarrow +\infty$

For large positive values of t which is larger: e^{-5t} or e^{-2t} ?

For the following problems, consider the case when $c_1 \neq 0$ and $c_2 \neq 0$ where the general solution is $\mathbf{x} = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-5t} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t}$,

For large positive values of t , which term dominates: $c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-5t}$ or $c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t}$?

Thus for large positive values of t , such trajectories (where $c_1 c_2 \neq 0$) when projected into the x_1, x_2 plane exhibit the following behavior (select all that apply):

* moves away from the origin.

* moves toward the origin.

* approaches the line $y = mx$ with slope $m = 3/-1$

* approaches a line $y = mx + b$ for $b \neq 0$ with slope $m =$. Note this case corresponds to where both $\|c_1 \mathbf{v}\| e^{r_1 t}$ and $\|c_2 \mathbf{w}\| e^{r_2 t}$ are large, but one is significantly larger than the other.

$t \rightarrow -\infty$ $t = 1000$ e^{5000} \gg e^{2000} large

For large negative values of t which is larger: e^{-5t} or e^{-2t} ?

For large negative values of t , which term dominates: $c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-5t}$ or $c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t}$?

Thus for large negative values of t , such trajectories (where $c_1 c_2 \neq 0$) when projected into the x_1, x_2 plane exhibit the following behavior (select all that apply):

* moves away from the origin.

* moves toward the origin.

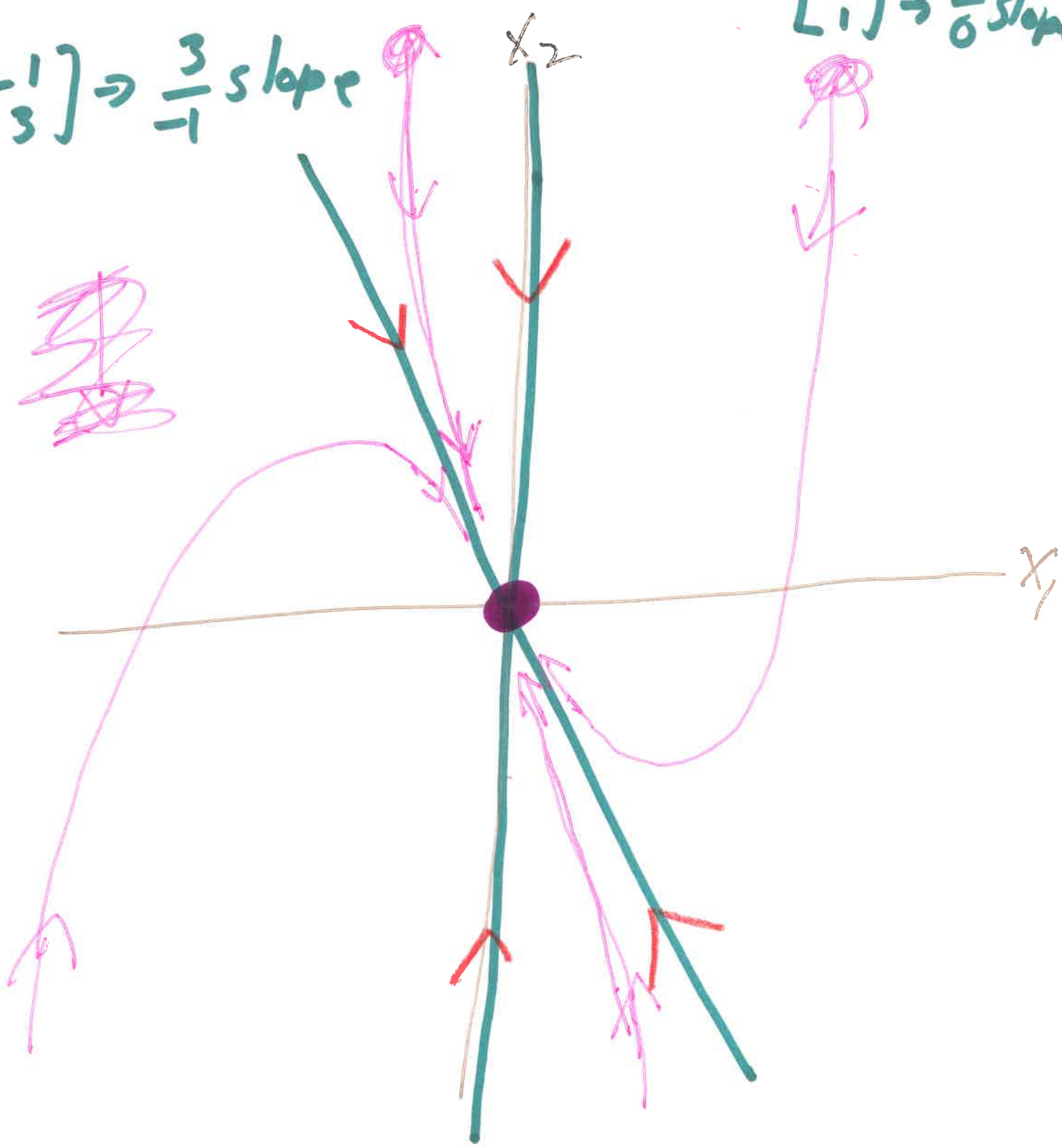
* approaches the line $y = mx$ with slope $m =$

* approaches a line $y = mx + b$ for $b \neq 0$ with slope $m =$. Note this case corresponds to where both $\|c_1 \mathbf{v}\| e^{r_1 t}$ and $\|c_2 \mathbf{w}\| e^{r_2 t}$ are large, but one is significantly larger than the other.

Two e. values

$[-1 \ 3] \rightarrow \frac{3}{-1}$ slope

$[0 \ 1] \rightarrow \frac{1}{0}$ slope



negative exponential go towards the origin

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 9 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x_1' = 2x_1$$

$$x_2' = 9x_1 + 5x_2$$

$$\frac{x_2'}{x_1'} = \frac{dx_2/dt}{dx_1/dt} = \boxed{\frac{dx_2}{dx_1} = \frac{9x_1 + 5x_2}{2x_1}}$$