Example 1: Given that the solution to to $\mathbf{x}' = \begin{bmatrix}-2 & 0\\ 21 & 5\end{bmatrix}\mathbf{x}$ is $\mathbf{x} = c_1\begin{bmatrix}-1\\ 3\end{bmatrix}e^{-2t} + c_2\begin{bmatrix}0\\ 1\end{bmatrix}e^{5t}$ arr

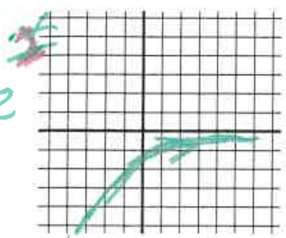
IVP: Plug in, solve for c_1 and c_2

$\Rightarrow c_1 = 1, \; c_2 = 0 \Rightarrow$ soln: $\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}-1\\ 3\end{bmatrix}e^{-2t}$

Graph the solution to the IVP $\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix} = \begin{bmatrix}-1\\ 3\end{bmatrix}$ in the

t, x_1-plane

$x_1 = -e^{-2t}$

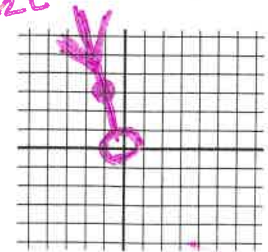


t, x_2-plane

$x_2 = 3e^{-2t}$

x_1, x_2-plane

$\frac{x_2}{x_1} = \frac{3\cancel{e^{-2t}}}{-\cancel{e^{-2t}}}$

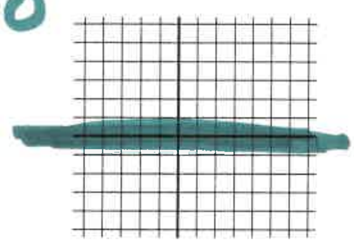


$x_2 = \frac{3}{-1}x_1$

Graph the solution to the IVP $\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$ in the

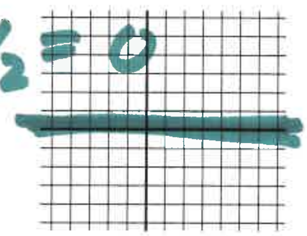
t, x_1-plane

$x_1 = 0$



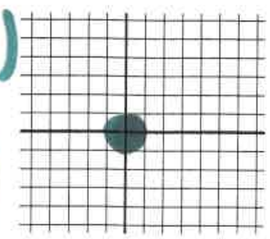
t, x_2-plane

$x_2 = 0$



x_1, x_2-plane

$(x_1, x_2) = (0, 0)$

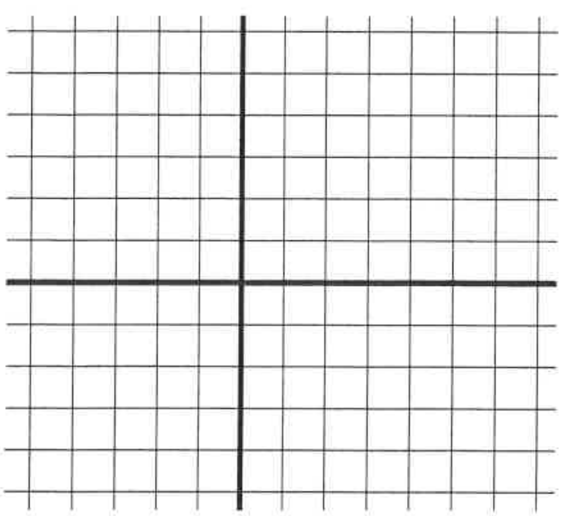


The equilibrium solution for this system of equations is $\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$.

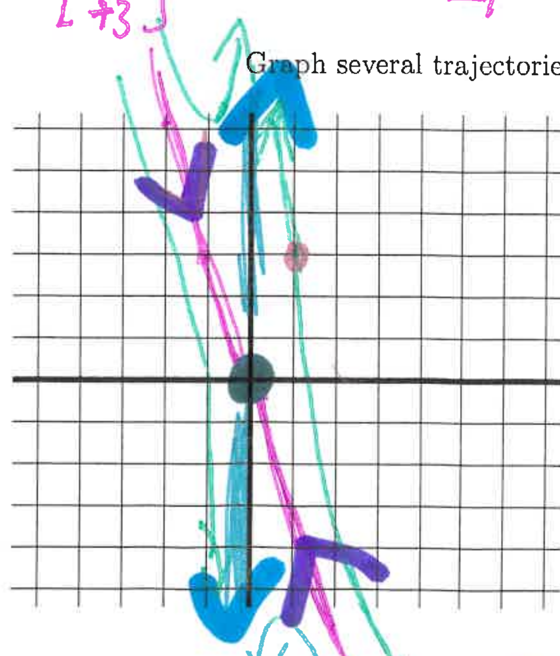
$\frac{dx_2}{dx_1} =$ ____________

$\begin{bmatrix}-1\\ +3\end{bmatrix} \rightarrow$ slope $\frac{3}{-1}$ ← neg exp

Plot several direction vectors where the slope is 0 and where slope is vertical.



Graph several trajectories.



pos exp $\rightarrow \begin{bmatrix}0\\ 1\end{bmatrix} \rightarrow$ slope $1/0$

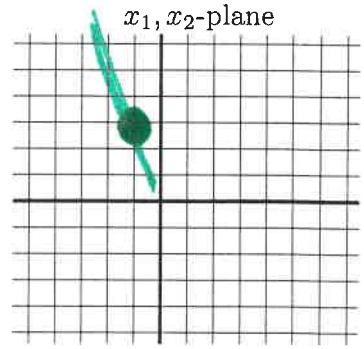
Semi-generic ex: Given that the solution to to $\mathbf{x}' = \begin{bmatrix} -2 & 0 \\ 21 & 5 \end{bmatrix} \mathbf{x}$ is $\mathbf{x} = c_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{r_1 t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{r_2 t}$

IVP: $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ implies $c_1 = 1, c_2 = 0$. Thus IVP soln: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{r_1 t}$

Hence $x_1 = -e^{r_1 t} < 0$ and $x_2 = 3e^{r_1 t} > 0$

and $\frac{x_2}{x_1} = \frac{3e^{r_1 t}}{-e^{r_1 t}} = \frac{3}{-1}$. Thus $x_2 = \frac{3}{-1} x_1$.

https://www.geogebra.org/3d (t, -e∧(-2t), 3*e∧(-2t))

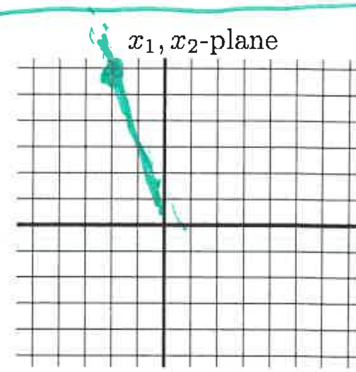


IVP: $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$ implies $c_1 = 2, c_2 = 0$. Thus IVP soln: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{r_1 t}$

Hence $x_1 = -2e^{r_1 t} < 0$ and $x_2 = 6e^{r_1 t} > 0$

and $\frac{x_2}{x_1} = \frac{6\cancel{e^{r_1 t}}}{-2\cancel{e^{r_1 t}}} = \frac{3}{-1}$. Thus $x_2 = \frac{3}{-1} x_1$.

https://www.geogebra.org/3d (t, -2*e∧(-2t), 6*e∧(-2t))

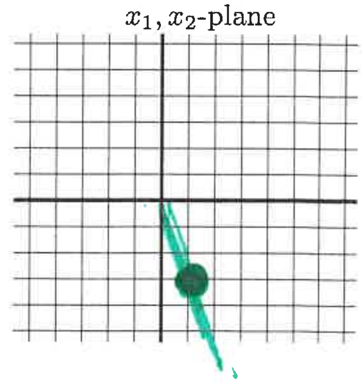


IVP: $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ implies $c_1 = -1, c_2 = 0$. Thus IVP soln: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = - \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{r_1 t}$

Hence $x_1 = e^{r_1 t} > 0$ and $x_2 = -3e^{r_1 t} < 0$

and $\frac{x_2}{x_1} = \frac{3e^{r_1 t}}{-e^{r_1 t}} = \frac{-3}{1}$. Thus $x_2 = \frac{-3}{1} x_1$.

https://www.geogebra.org/3d (t, e∧(-2t), -3*e∧(-2t))



Semi-generic ex: Given that the solution to to $\mathbf{x}' = \begin{bmatrix} -2 & 0 \\ 21 & 5 \end{bmatrix} \mathbf{x}$ is $\mathbf{x} = c_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{r_2 t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{r_2 t}$

IVP: $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ implies $c_1 = 0, c_2 = 1$.

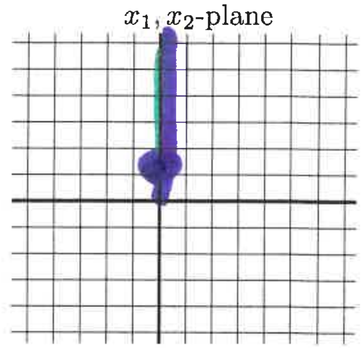
Thus IVP soln: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{r_2 t}$

Hence $x_1 = 0$ and $x_2 = e^{r_2 t} > 0$

and $\frac{x_2}{x_1} = \frac{1e^{r_2 t}}{0e^{r_2 t}} = \frac{1}{0}$. Thus $x_2 = \frac{1}{0} x_1$.

https://www.geogebra.org/3d (t, 0, e∧(5t))

x_1, x_2-plane



IVP: $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ implies $c_1 = 0, c_2 = -1$.

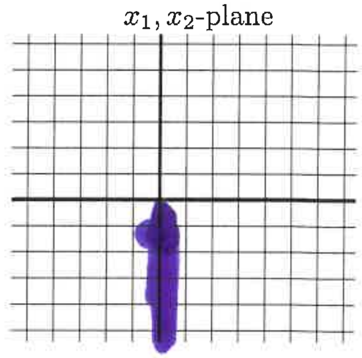
Thus IVP soln: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -\begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{r_2 t}$

Hence $x_1 = 0$ and $x_2 = -e^{r_2 t} < 0$

and $\frac{x_2}{x_1} = \frac{-1e^{r_2 t}}{0e^{r_2 t}} = \frac{-1}{0}$. Thus $x_2 = \frac{-1}{0} x_1$.

https://www.geogebra.org/3d (t, 0, -e∧(5t))

x_1, x_2-plane



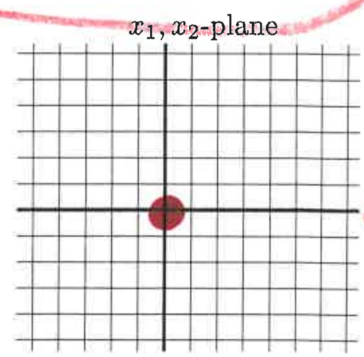
IVP: $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ implies $c_1 = c_2 = 0$.

Thus IVP soln: $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Hence $x_1 = 0$ and $x_2 = 0$

https://www.geogebra.org/3d (t, 0, 0)

x_1, x_2-plane



$\left.\begin{array}{l} x_1(t) = 0 \\ x_2(t) = 0 \end{array}\right\}$ constant soln

=

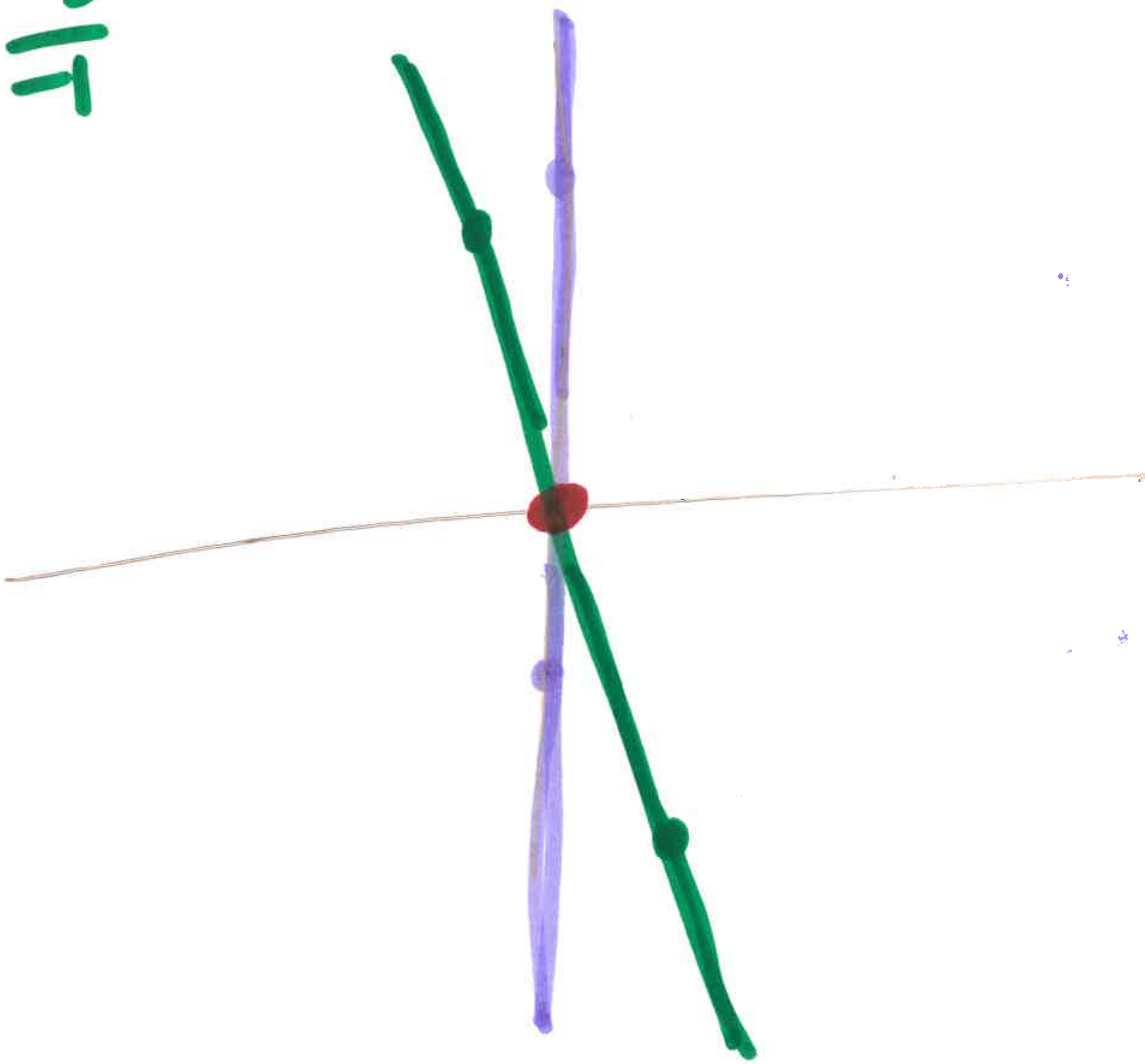
equilibrium soln

$\vec{x}' = A\vec{x} \Rightarrow \vec{x}(t) = 0$ is a soln $\Rightarrow \vec{x}(t)$ is an equil soln

$$X = c_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{r_1 t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{r_2 t}$$

$\frac{3}{-1}$

$1/0$



Example 2: Given that the solution to $\mathbf{x}' = \begin{bmatrix} -2 & 0 \\ -9 & -5 \end{bmatrix}\mathbf{x}$ is $\mathbf{x} = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-5t} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t}$

negative exponential

$\vec{x}(t) \to (0,0)$

Graph the solution to the IVP $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ in the

t, x_1-plane

t, x_2-plane

x_1, x_2-plane

Graph the solution to the IVP $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in the

t, x_1-plane

t, x_2-plane

x_1, x_2-plane

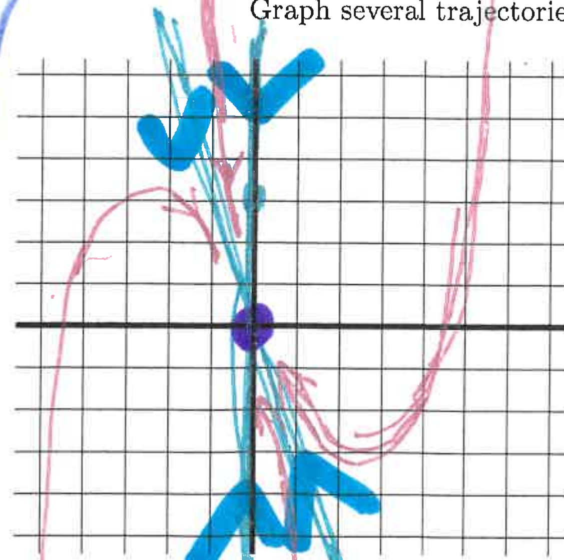
The equilibrium solution for this system of equations is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \ \end{bmatrix}$.

$\frac{dx_2}{dx_1} =$ ____________

$\begin{bmatrix} -1 \\ 3 \end{bmatrix} \to \frac{3}{-1}$

Plot several direction vectors where the slope is 0 and where slope is vertical.

Graph several trajectories.



Corresponds to negative e.values = negative exponentials

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \to 1/0$

Example 3: Given that the solution to $\mathbf{x}' = \begin{bmatrix} 2 & 0 \\ 9 & 5 \end{bmatrix} \mathbf{x}$ is $\mathbf{x} = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{2t}$

e. values > 0

exponentials

Graph the solution to the IVP $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ in the

t, x_1-plane $\qquad$ t, x_2-plane $\qquad$ x_1, x_2-plane

Graph the solution to the IVP $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in the

t, x_1-plane $\qquad$ t, x_2-plane $\qquad$ x_1, x_2-plane

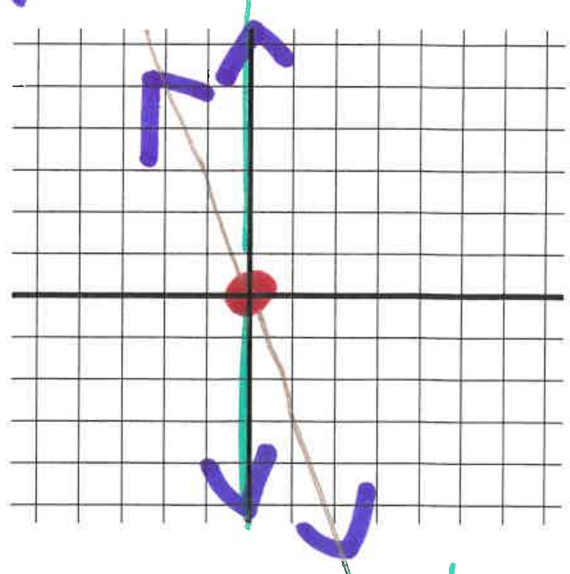
The equilibrium solution for this system of equations is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$\frac{dx_2}{dx_1} =$ ____________

$\begin{bmatrix} -1 \\ 3 \end{bmatrix} \to$ slope $\frac{3}{-1}$

Plot several direction vectors where the slope is 0 and where slope is vertical.

Graph several trajectories.



positive exp

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \to$ slope $\frac{1}{0}$

$\begin{bmatrix}-1\\3\end{bmatrix}$ $c_1 \neq 0$

Answer the following questions for $A = \begin{bmatrix} -2 & 0 \\ 21 & 5 \end{bmatrix}$:

The smaller eigenvalue of A is $r_1 =$ -2. An eigenvector corresponding to r_1 is $\mathbf{v} =$ $c_2 \neq 0$

The larger eigenvalue of A is $r_2 =$ 5. An eigenvector corresponding to r_2 is $\mathbf{w} =$ $\begin{bmatrix}0\\1\end{bmatrix}$

The general solution to $\mathbf{x}' = \begin{bmatrix} -2 & 0 \\ 21 & 5 \end{bmatrix}\mathbf{x}$ is $\boxed{\mathbf{x} = c_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t}}$

$t \to +\infty$ $e^{-2t} \to 0$

$e^{5t} \to \infty$

For large **positive** values of t which is larger: e^{-2t} or e^{5t}?

For the following problems, consider the case when $c_1 \neq 0$ and $c_2 \neq 0$.

For large **positive** values of t, which term dominates: $\underbrace{c_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t}}_{\text{tiny}}$ or $c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t}$ dominate

Thus for large **positive** values of t, such trajectories (where $c_1 c_2 \neq 0$) when projected into the x_1, x_2 plane exhibit the following behavior (select all that apply):

* moves away from the origin. positive exp $\begin{bmatrix}0\\1\end{bmatrix}$
* moves toward the origin.
* approaches the line $y = mx$ with slope $m =$ $1/0$
* approaches a line $y = mx + b$ for $b \neq 0$ with slope $m =$ __________. Note this case corresponds to where both $||c_1\mathbf{v}||e^{r_1 t}$ and $||c_2\mathbf{w}||e^{r_2 t}$ are large, but one is significantly larger than the other.

$t = -1000$ $e^{2000} \gg e^{-5000}$ $t \to -\infty$ $e^{-2t} \to \infty$

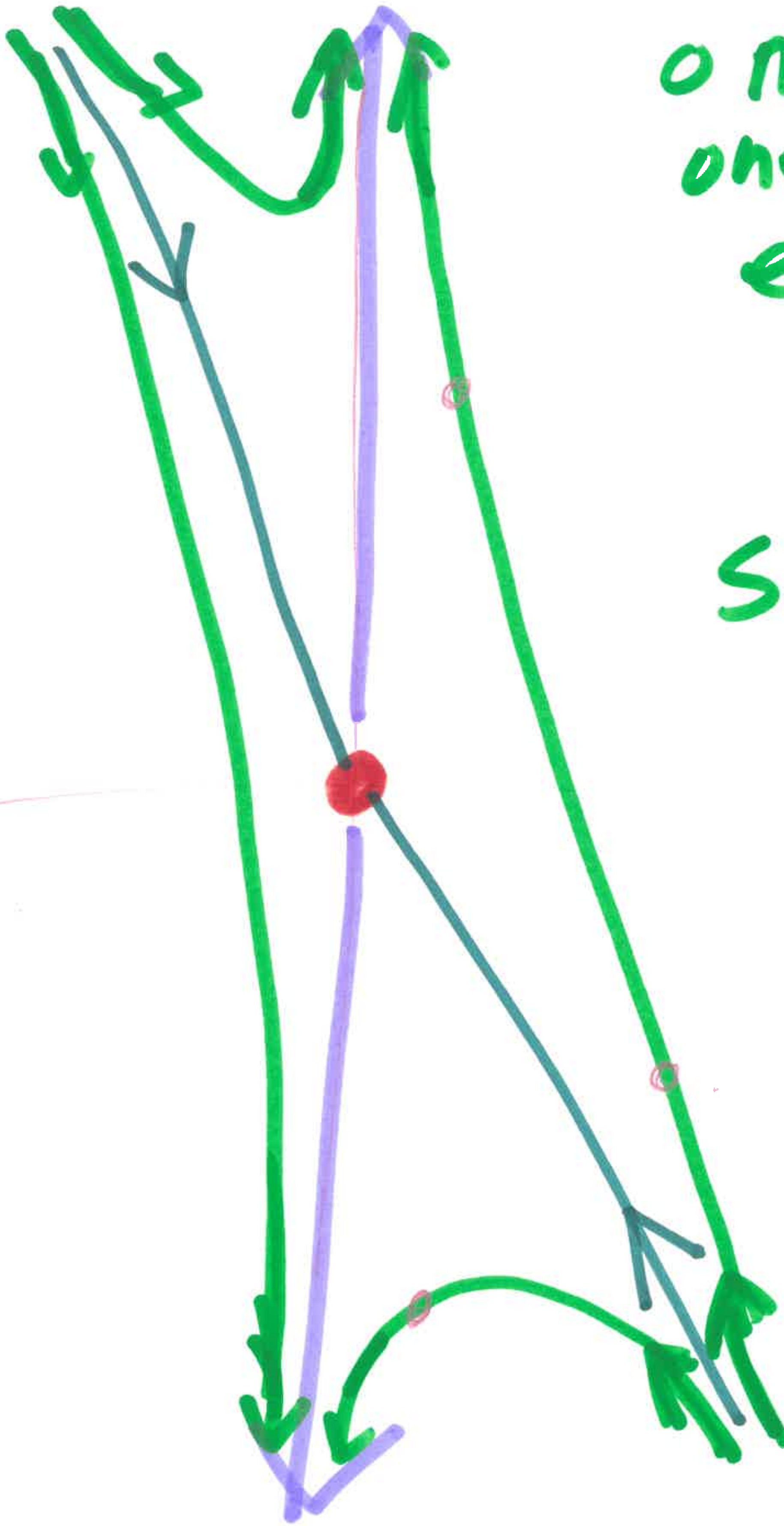
$e^{5t} \to 0$

For large **negative** values of t which is larger: e^{-2t} or e^{5t}?

For large **negative** values of t, which term dominates: $c_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t}$ or $\underbrace{c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t}}_{\text{tiny}}$?

Thus for large **negative** values of t, such trajectories (where $c_1 c_2 \neq 0$) when projected into the x_1, x_2 plane exhibit the following behavior (select all that apply):

* ~~moves away from the origin.~~ backwards in time
* ~~moves toward the origin.~~
* approaches the line $y = mx$ with slope $m =$ $3/-1$
* approaches a line $y = mx + b$ for $b \neq 0$ with slope $m =$ __________. Note this case corresponds to where both $||c_1\mathbf{v}||e^{r_1 t}$ and $||c_2\mathbf{w}||e^{r_2 t}$ are large, but one is significantly larger than the other.



one +
one −
e. value
$\Downarrow$
saddle

Answer the following questions for $A = \begin{bmatrix} 2 & 0 \\ 9 & 5 \end{bmatrix}$:

The smaller eigenvalue of A is $r_1 =$ 2. An eigenvector corresponding to r_1 is $\mathbf{v} =$ $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

The larger eigenvalue of A is $r_2 =$ 5. An eigenvector corresponding to r_2 is $\mathbf{w} =$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

The general solution to $\mathbf{x} = \begin{bmatrix} 2 & 0 \\ 9 & 5 \end{bmatrix}\mathbf{x}$ is $\mathbf{x} = c_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t}$

two positive e.values

$t = 1000 \quad e^{2000} << e^{5000}$

For large **positive** values of t which is larger: e^{2t} or e^{5t}?

For the following problems, consider the case when $c_1 \neq 0$ and $c_2 \neq 0$ where the general solution is $\mathbf{x} = c_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t}$

For large **positive** values of t, which term dominates: $c_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{2t}$ or $c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t}$

large   very very large

Thus for large **positive** values of t, such trajectories (where $c_1 c_2 \neq 0$) when projected into the x_1, x_2 plane exhibit the following behavior (select all that apply):

* moves away from the origin. ← pos exp
* moves toward the origin.
* approaches the line $y = mx$ with slope $m =$ ________
* approaches a line $y = mx + b$ for $b \neq 0$ with slope $m =$ 1/0. Note this case corresponds to where both $||c_1\mathbf{v}||e^{r_1 t}$ and $||c_2\mathbf{w}||e^{r_2 t}$ are large, but one is significantly larger than the other.

$t = -1000 \quad e^{-2000} >> e^{-5000}$

For large **negative** values of t which is larger: e^{2t} or e^{5t}?

For large **negative** values of t, which term dominates: $c_1 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{2t}$ or $c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t}$

small   tiny

Thus for large **negative** values of t, such trajectories (where $c_1 c_2 \neq 0$) when projected into the x_1, x_2 plane exhibit the following behavior (select all that apply):

* ~~moves away from the origin.~~
* ~~moves toward the origin.~~
* approaches the line $y = mx$ with slope $m =$ 3/-1
* approaches a line $y = mx + b$ for $b \neq 0$ with slope $m =$ ________. Note this case corresponds to where both $||c_1\mathbf{v}||e^{r_1 t}$ and $||c_2\mathbf{w}||e^{r_2 t}$ are large, but one is significantly larger than the other.

Answer the following questions for $A = \begin{bmatrix} -2 & 0 \\ -9 & -5 \end{bmatrix}$:

The smaller eigenvalue of A is $r_1 =$ $\underline{-5}$. An eigenvector corresponding to r_1 is $\mathbf{v} =$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

The larger eigenvalue of A is $r_2 =$ $\underline{-2}$. An eigenvector corresponding to r_2 is $\mathbf{w} =$ $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

The general solution to $\mathbf{x}' = \begin{bmatrix} -2 & 0 \\ -9 & -5 \end{bmatrix} \mathbf{x}$ is $\mathbf{x} = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-5t} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t}$

$t = 1000$ $\quad e^{-5000} < e^{-2000}$ ← larger

For large **positive** values of t which is larger: e^{-5t} or e^{-2t}?

For the following problems, consider the case when $c_1 \neq 0$ and $c_2 \neq 0$ where the general solution is $\mathbf{x} = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-5t} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t}$,

For large **positive** values of t, which term dominates: $c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-5t}$ or $c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t}$?

very small $\qquad$ small

Thus for large **positive** values of t, such trajectories (where $c_1 c_2 \neq 0$) when projected into the x_1, x_2 plane exhibit the following behavior (select all that apply):

* moves away from the origin.
* moves toward the origin.
* approaches the line $y = mx$ with slope $m =$ $\underline{3/-1}$
* approaches a line $y = mx + b$ for $b \neq 0$ with slope $m =$ __________. Note this case corresponds to where both $||c_1 \mathbf{v}|| e^{r_1 t}$ and $||c_2 \mathbf{w}|| e^{r_2 t}$ are large, but one is significantly larger than the other.

$t = 1000$ $\quad e^{5000} >> e^{2000}$

large

For large **negative** values of t which is larger: e^{-5t} or e^{-2t}?

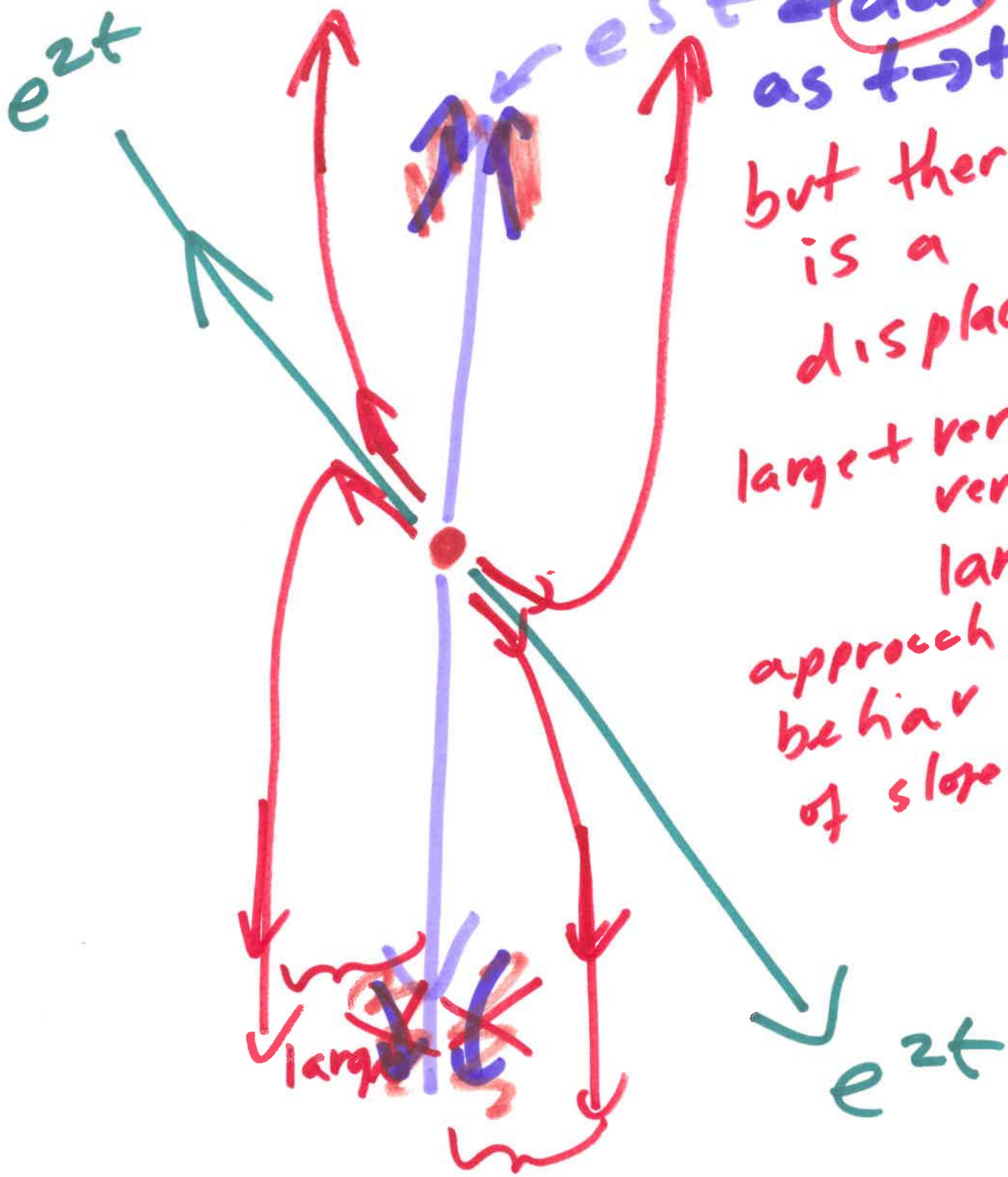
For large **negative** values of t, which term dominates: $c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-5t}$ or $c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} e^{-2t}$?

Thus for large **negative** values of t, such trajectories (where $c_1 c_2 \neq 0$) when projected into the x_1, x_2 plane exhibit the following behavior (select all that apply):

very large $\qquad$ large

* moves away from the origin.
* ~~moves toward the origin.~~
* approaches the line $y = mx$ with slope $m =$ __________
* approaches a line $y = mx + b$ for $b \neq 0$ with slope $m =$ $\underline{1/0}$. Note this case corresponds to where both $||c_1 \mathbf{v}|| e^{r_1 t}$ and $||c_2 \mathbf{w}|| e^{r_2 t}$ are large, but one is significantly larger than the other.

Two + e. values



but there is a displaced large + very very large approach behav of slope $\frac{1}{0}$

large