To find $\mathcal{L}^{-1}$ = inverse LaPlace

The algebra techniques that you will use in ch 6 include

partial fractions,
completing the square,
adding 0, and
multiplying by 1.

1. Look at denominator s^2+2s+5

i.) Can you factor denominator over the reals? If so, factor and use partial fractions.

ii.) Does your denominator equal one of the following? s^n, $s-a$, $s^2 \pm a^2$, If so, use the appropriate formula.

iii.) Do you need to complete the square? Example: when Does not factor over $\mathbb{R}$

$$\begin{aligned} 10s^2+60s+91 &= 10(s^2+6s)+91 \\ &= 10(s^2+6+9-9)+91 \\ &= 10(s^2+6+9)-90+91 \\ &= 10(s+3)^2+1 \end{aligned}$$

ie quadratic w/ complex roots

2. Look at the numerator

i.) Do you need $s-a$? Try adding 0. For example to make $s+\frac{3}{2}$ appear in $5s+21$:

$$5s+21 = 5(s+\tfrac{3}{2}) - \tfrac{15}{2} + 21 = 5(s+\tfrac{3}{2}) + \tfrac{27}{2}$$

ii) Do you need b? Try multiplying by 1. For example, if you need $\sqrt{\frac{7}{4}}$, but you have $\frac{27}{2}$: $\quad \frac{27}{2} = \frac{27}{2}\sqrt{\frac{4}{7}}\sqrt{\frac{7}{4}}$

TABLE 6.2.1 Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	Notes
1. 1	$\dfrac{1}{s}, \quad s > 0$	Sec. 6.1; Ex. 4
2. e^{at}	$\dfrac{1}{s-a}, \quad s > a$	Sec. 6.1; Ex. 5
3. $t^n, \quad n =$ positive integer	$\dfrac{n!}{s^{n+1}}, \quad s > 0$	Sec. 6.1; Prob. 31
4. $t^p, \quad p > -1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$	Sec. 6.1; Prob. 31
5. $\sin at$	$\dfrac{a}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Ex. 7
6. $\cos at$	$\dfrac{s}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Prob. 6
7. $\sinh at$	$\dfrac{a}{s^2-a^2}, \quad s > \|a\|$	Sec. 6.1; Prob. 8
8. $\cosh at$	$\dfrac{s}{s^2-a^2}, \quad s > \|a\|$	Sec. 6.1; Prob. 7
9. $e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 13
10. $e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 14
11. $t^n e^{at}, \quad n =$ positive integer	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$	Sec. 6.1; Prob. 18
12. $u_c(t)$	$\dfrac{e^{-cs}}{s}, \quad s > 0$	Sec. 6.3
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$	Sec. 6.3
14. $e^{ct}f(t)$	$F(s-c)$	Sec. 6.3
15. $f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right), \quad c > 0$	Sec. 6.3; Prob. 25
16. $\displaystyle\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	Sec. 6.6
17. $\delta(t-c)$	e^{-cs}	Sec. 6.5
18. $f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	Sec. 6.2; Cor. 6.2.2
19. $(-t)^n f(t)$	$F^{(n)}(s)$	Sec. 6.2; Prob. 29

Only use If quadratic does not factor over $\mathbb{R}$

ie $(s-a)^2+b^2=0$ has complex roots

Solve (i.e., solve for y):

Ch 3: plug in eqn $r^4 = 0$

char eqn

$y^{iv} = 0, \; y(0) = 0, \; y'(0) = 0, \; y''(0) = 0, \; y'''(0) = 10$

$\mathcal{L}(y^{iv}) = \mathcal{L}(0)$

$s^4\mathcal{L}(y) - s^3 10y(0) - s^2y'(0) - sy''(0) - y'''(0) = 0$

$s^4\mathcal{L}(y) - 10 = 0$

$s^4\mathcal{L}(y) = 10$

characteristic polynom f(s)

$\mathcal{L}(y) = \frac{10}{s^4}$

$\mathcal{L}^{-1}(\mathcal{L}(y)) = \mathcal{L}^{-1}(\frac{10}{s^4})$

$y = \mathcal{L}^{-1}(\frac{10}{s^4})$ Formula 3: $\mathcal{L}^{-1}(\frac{n!}{s^{n+1}}) = t^n$

$y = \mathcal{L}^{-1}(\frac{10}{3!} \cdot \frac{3!}{s^4}) = \frac{10}{3!}\mathcal{L}^{-1}(\cdot\frac{3!}{s^4}) = \frac{10}{3!}t^3 = \frac{5}{3}t^3$

Solve (i.e., solve for y):

characteristic eqn

$y'' + 2y' + 5y = 0,\ y(0) = 1,\ y'(0) = 7$

$\mathcal{L}(y'' + 2y' + 5y) = \mathcal{L}(0)$

$\mathcal{L}(y'') + 2\mathcal{L}(y') + 5\mathcal{L}(y) = 0$

$s^2\mathcal{L}(y) - sy(0) - y'(0)$

$+2[s\mathcal{L}(y) - y(0)]$

$+5\mathcal{L}(y) = 0$

$s^2\mathcal{L}(y) - s - 7$

$+2[s\mathcal{L}(y) - 1]$

$+5\mathcal{L}(y) = 0$

$s^2\mathcal{L}(y) - s - 7$

$+2s\mathcal{L}(y) - 2$

$+5\mathcal{L}(y) = 0$

characteristic poly

$(s^2 + 2s + 5)\mathcal{L}(y) = s + 7 + 2$

$\mathcal{L}(y) = \frac{s+9}{s^2+2s+5}$

$\mathcal{L}^{-1}(\mathcal{L}(y)) = \mathcal{L}^{-1}(\frac{s+9}{s^2+2s+5})$

$b^2 - 4ac$

$4 - 4(1)(5) < 0$

$y = \mathcal{L}^{-1}(\frac{s+9}{s^2+2s+5})$

complete the square ← complex roots

$$y = \mathcal{L}^{-1}(\frac{s+9}{s^2+2s+1-1+5})$$

$$y = \mathcal{L}^{-1}(\frac{s+9}{(s+1)^2-1+5})$$

$$y = \mathcal{L}^{-1}(\frac{s+9}{(s+1)^2+4})$$

$$y = \mathcal{L}^{-1}(\frac{s+1+8}{(s+1)^2+4})$$

$$y = \mathcal{L}^{-1}(\frac{s+1}{(s+1)^2+4} + \frac{8}{(s+1)^2+4})$$

$$y = \mathcal{L}^{-1}(\frac{s+1}{(s+1)^2+4}) + \mathcal{L}^{-1}(\frac{8}{(s+1)^2+4})$$

$$y = \mathcal{L}^{-1}(\frac{s+1}{(s+1)^2+4}) + 4\mathcal{L}^{-1}(\frac{2}{(s+1)^2+4})$$

$$y = \underline{e^{-t}cos(2t)} + 4\underline{e^{-t}sin(2t)}$$

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compare to ch 3/4 method

Partial Check: Plug $\underline{y = e^{rt}}$ into $y'' + 2y' + 5y = 0$:

Characteristic eqn: $r^2 + 2r + 5 = 0$ implies

$$(r+1)^2 + 4 = 0 \text{ and thus } (r+1)^2 = -4.$$

Hence $r + 1 = \sqrt{-4} = \pm 2i$. Thus $r = -1 \pm 2i$

and the general homogeneous soln is

$$y = c_1\underline{e^{-t}cos(2t)} + c_2\underline{e^{-t}sin(2t)}$$

Partial Check:

$\underline{y(0) = 1}$ for IVP soln $y = e^{-t}cos(2t) + 4e^{-t}sin(2t)$

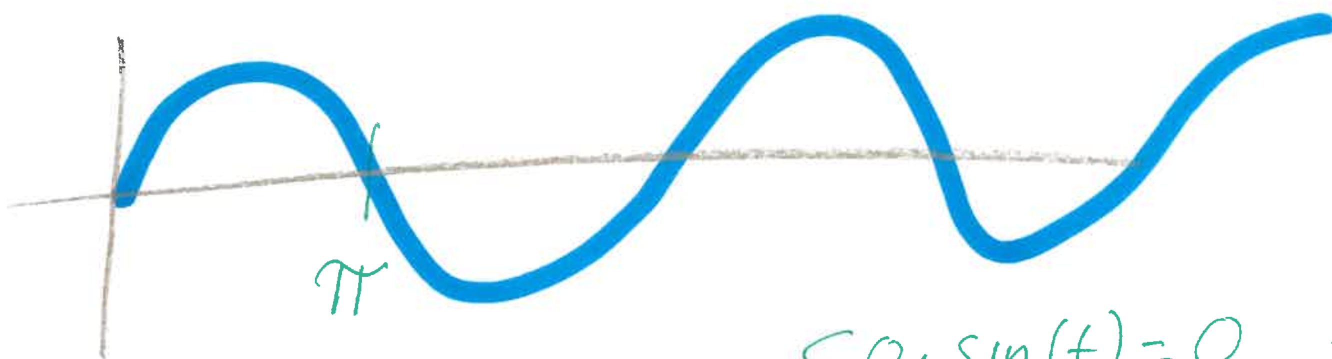
$1 = e^0cos(0) + 4e^0sin(0)$ ✓ where equality does hold

Partial Check: $\underline{y'(0) = 7}$ - too much work due to product rule.

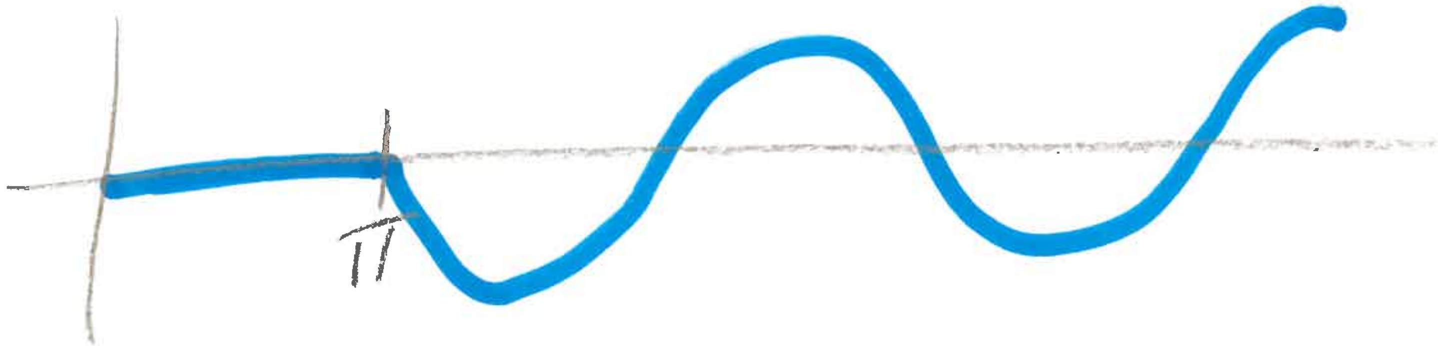
6.3: Step functions.

Graph $u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$

Graph $g(t) = sin(t)$.



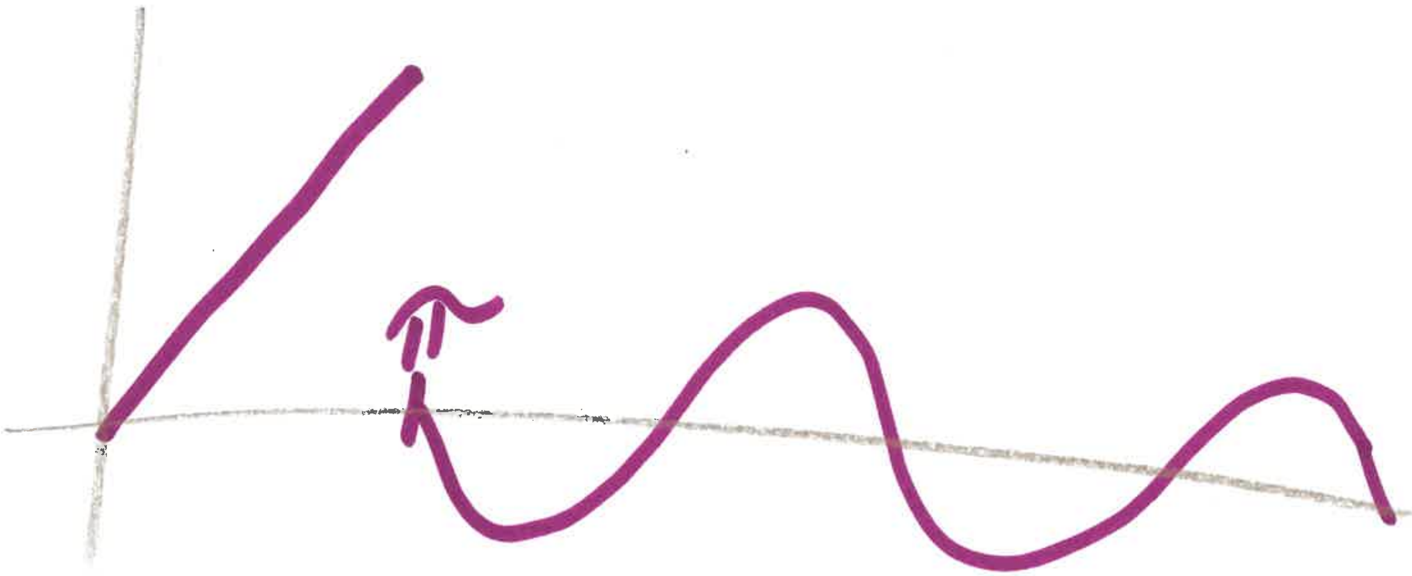
Graph $h(t) = u_\pi(t) sin(t)$. $= \begin{cases} 0 \cdot \sin(t) = 0 & t < \pi \\ 1 \cdot \sin t = \sin t & t > \pi \end{cases}$



$t < \pi$, $f(t) = 2t + 0[\;] = 2t$

Graph $f(t) = 2t + u_\pi(t)[sin(t) - 2t] = \begin{cases} 2t & t < \pi \\ sin(t) & t \geq \pi \end{cases}$

$t > \pi$ $\quad \cancel{2t} + 1\,[sin(t) - \cancel{2t}]$



$$h(t) = \begin{cases} t & 0 \leq t < 4 \\ ln(t) & t \geq 4 \end{cases}$$

implies $h(t) =$