Section:____________ **Name:**____________________________________

MATH:2560 Engineer Math IV: Differential Equations

MIDTERM ONE EXAMINATION

September 28, 2023

This examination booklet contains 4 problems (with 4th problem consisting of multiple parts) worth a total of 50 points on 6 sheets of paper including the front cover and a blank page on the back.

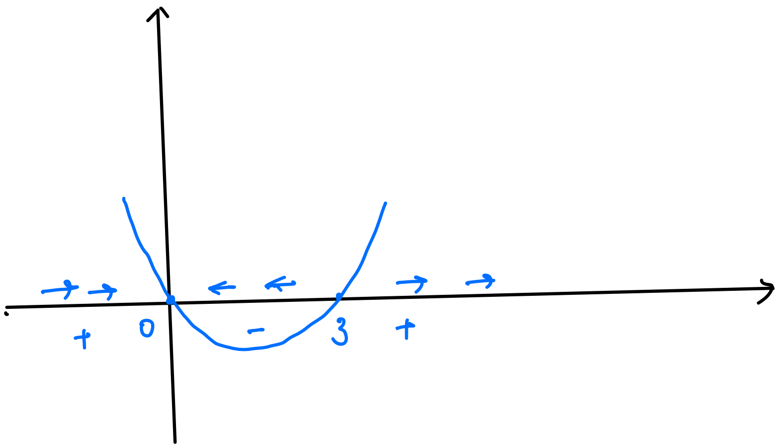
Do all of your work in this booklet and show all your computations and clearly indicate your answers.

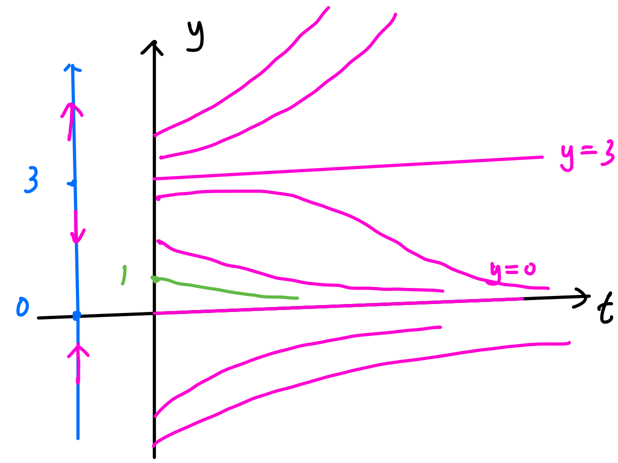
No calculators, phones, ipads, smart watches or any other internet accessing devices are allowed during the exam time. This exam is closed book and notes. Keep your notes, books, electronic devices in your backpack at ALL times during the exam period.

Problem	Points	Score
1	10	
2	10	
3	10	
4	20	
Total	50	

1. Consider the differential equation $y' = y(y-3)$.

1a. Draw the phase line and sketch several graphs in the ty-plane of solutions to the differential equation $y' = y(y-3)$. Include graphs of the equilibrium solutions as well as trajectories that are above, below, and in between the equilibrium solutions.





1b. State the equilibrium solutions and determine their stability.

$y =$ ___0___ is an ___stable___ equilibrium solution.

$y =$ ___3___ is an ___unstable___ equilibrium solution.

If $y(t)$ is a solution to the initial value problem $y' = y(y-3),\ y(0) = 1$, then $\lim_{t\to+\infty} y(t) =$ ___0___

1b. For the initial value problem $y' = y(y-3),\ y(0) = 1$, use Eulers method with a step size of $\Delta t = 0.1$ to estimate $y(0.2)$

$h=$

$=$

$y(0.2) \approx$ ______________

Sol: Note the initial point is $(t_0, y_0) = (0, 1)$. The iteration formula is

$$\begin{cases} y_0 = 1,\ t_0 = 0; \\ y_{n+1} = f(t_n, y_n)\cdot h = y_n(y_n - 3)\cdot 0.1; \quad n \geq 0 \end{cases}$$

$$\Rightarrow \quad y(0.1) = y_1 = y_0(y_0-3)\cdot 0.1 = -0.2$$

$$y(0.2) = y_2 = y_1(y_1-3)\cdot 0.1 = (-0.2)(-0.2-3)\cdot 0.1 = 0.2\cdot 3.2\cdot 0.1$$
$$= 0.064$$

Change to $\frac{dy}{dt}$

2. Find the general solution for the autonomous equation: $y' = y(y-3)$.

$$\frac{dy}{dt} = y(y-3)$$

$$* \quad \frac{1}{y(y-3)}\, dy = dt$$

$$\frac{1}{y(y-3)} = \frac{a}{y} + \frac{b}{y-3} = \frac{ay-3a+by}{y(y-3)}$$

$$= \frac{(a+b)y-3a}{y(y-3)}$$

$$-3a=1, \quad a+b=0 \Rightarrow a=-\frac{1}{3}, \; b=\frac{1}{3}$$

$$\frac{1}{y(y-3)} = -\frac{1}{3}\frac{1}{y} + \frac{1}{3}\frac{1}{y-3} = -\frac{1}{3}\left(\frac{1}{y} - \frac{1}{y-3}\right)$$

With this, eqn (*) is rewritten as

$$\left(\frac{1}{y} - \frac{1}{y-3}\right) dy = -3\,dt$$

Integrating:
$$\ln|y| - \ln|y-3| = -3t + C$$

$$\ln\left|\frac{y}{y-3}\right| = -3t + C$$

$$\left|\frac{y}{y-3}\right| = e^{-3t} e^{C} \qquad \frac{y}{y-3} = \pm e^{C} e^{-3t}$$

Hence $y=0$ is also a solution, so

$$\frac{y}{y-3} = c e^{-3t}, \qquad y = ce^{-3t}(y-3)$$

$$y = ce^{-3t}y - 3ce^{-3t}$$

$$y(1-ce^{-3t}) = -3ce^{-3t}$$

Answer: $y(t) = \dfrac{-3ce^{-3t}}{1-ce^{-3t}}$.

3. Solve the initial value problem: $ty' + y = \sin t, \quad y(\frac{\pi}{2}) = 1.$ (solve on $t>0$).

Sol: Dividing by t:

$$y' + \frac{1}{t} y = \frac{\sin t}{t}.$$

Integrating factor is

$$\mu(t) = e^{\int \frac{1}{t}\,dt} = e^{\ln t} = t.$$

So the general sol is

$$y(t) = \frac{1}{t} \int t \cdot \frac{\sin t}{t}\,dt$$

$$= \frac{1}{t} \int \sin t\,dt$$

$$= \frac{1}{t} (-\cos t + C)$$

$$y(\tfrac{\pi}{2}) = \frac{1}{\pi/2} C$$

$$y(\tfrac{\pi}{2}) = 1 \quad \Rightarrow C = \frac{\pi}{2}.$$

$$y(t) = \frac{1}{t}(-\cos t + \frac{\pi}{2}).$$

Answer: $y(t) = \frac{1}{t}(-\cos t + \frac{\pi}{2}).$

4. Fill in the blank:

a. A 50 gallon tank contains 3 grams of Kryptonite. 8 gallons of water containing 2 grams of Krypotonite is pumped into the tanke every minute, and 8 gallons of solution is drained from the tank every minute. Assuming a thorough mixing, setup the initial value problem that describes this process. **(Do not solve!)**

$Q(t)$: the amount of kryptonite at any time t.

$dQ(t)/dt$ = rate in - rate out. rate in = $2\,g/min$; rate out = $\frac{Q}{50}\cdot 8 = \frac{4}{25}Q$

1 1 1

Diff. eqn: $\frac{dQ}{dt} = 2 - \frac{4}{25}Q$ 1. Initial Value: $Q(0)=3$.

b. Calculate the Wronskian $W(t^{-\frac{1}{2}}, t^2) =$ $\frac{5}{2}t^{\frac{1}{2}}$.

$$W = \begin{vmatrix} t^{-\frac{1}{2}} & t^2 \\ -\frac{1}{2}t^{-\frac{3}{2}} & 2t \end{vmatrix} = 2t^{\frac{1}{2}} + \frac{1}{2}t^{\frac{1}{2}} = \frac{5}{2}t^{\frac{1}{2}}.$$

2 2

c. Give the **form** of the particular (nonhomogenous) solution with undetermined coefficients for

$$y'' + 5y' + 4y = 2\cos(t)$$

(1 pt for $A\cos t$ or $A\sin t$)

$Y(t) =$ $A\cos t + B\sin t$ 4 (Do not solve!)

d. The general solution for $y'' - y = 0$ is $c_1 e^t + c_2 e^{-t}$ 4

$r^2 - 1 = 0 \quad r = \pm 1.$

3.5 pts for $\{e^t, e^{-t}\}$.

e. The general solution for $y'' + 9y = 0$ is $c_1\cos 3t + c_2\sin 3t$.

4

3.5 pts for $\{\cos 3t, \sin 3t\}$.