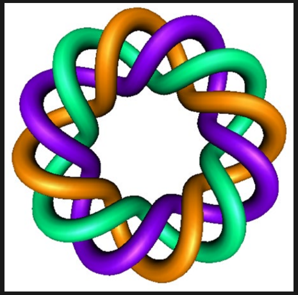
2.4 Existence and Uniqueness



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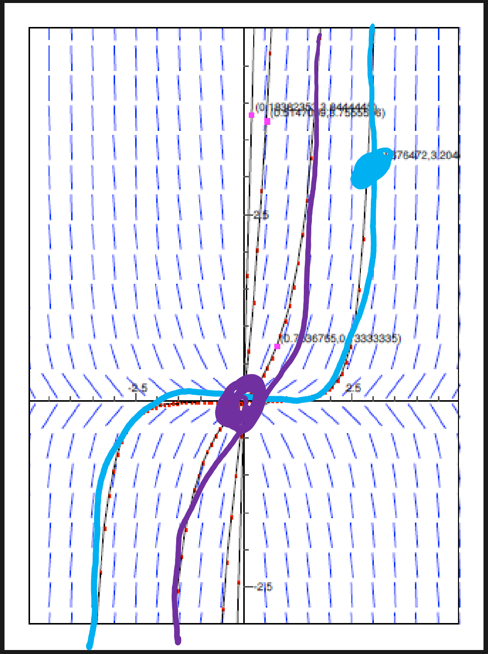
Initial value: A chosen point (t_0, y_0) through which a solution must pass. I.e., (t_0, y_0) lies on the graph of the solution that satisfies this initial value.



Initial value problem (IVP): A differential equation where initial value is specified.

An initial value problem can have 0, 1, or multiple equilibrium solutions (finite or infinite).

Section 2.4: Existence and Uniqueness.

In general, for IVP $\boxed{y' = f(t, y),\ y(t_0) = y_0}$ solution may or may not exist and solution may or may not be unique.

1.) Does there exist a solution to the DE $y' = f(t, y)$?

often need to solve numerically (approx soln)

real world

- If no, then solution to IVP $y' = f(t, y)$, $y(t_0) = y_0$ does not exist.

  Numerical answer is wrong if soln does not exist

- If yes, then general solution to DE $y' = f(t, y)$ will include a constant term C

  could be more than a single eqn

2.) If solution exists to DE $y' = f(t, y)$, then does there exist a solution for C to IVP, $y(t_0) = y_0$ and is that solution unique.

our class

Examples: No solution:

Ex 1: $\cancel{y'} = \cancel{y} + 1$ $\Rightarrow$ $0 \neq 1$ $\forall x$

no soln

Ex 2: $(y')^2 = -1$ $\Longrightarrow$ no real-valued soln

positive#n ≠ negative

$f: \mathbb{R} \to \mathbb{R}$

Ex 3a (IVP): $\boxed{\frac{dy}{dx} = y(1 + \frac{1}{x}),\ y(0) = 1}$

$$\int \frac{dy}{y} = \int (1 + \frac{1}{x})dx \implies ln|y| = x + ln|x| + C$$

$$|y| = e^{x + ln|x| + C} = e^x e^{ln|x|} e^C = Cxe^x$$

$\boxed{\text{Thus general solution is } y = Cxe^x}$

$y(0) = 1$: $1 = C(0)e^0 = 0$ implies $1 \neq 0$

IVP $\frac{dy}{dx} = y(1 + \frac{1}{x})$, $y(0) = 1$ has no solution.

Ex 3a (IVP): $\frac{dy}{dx} = y(1 + \frac{1}{x})$, $y(0) = 1$

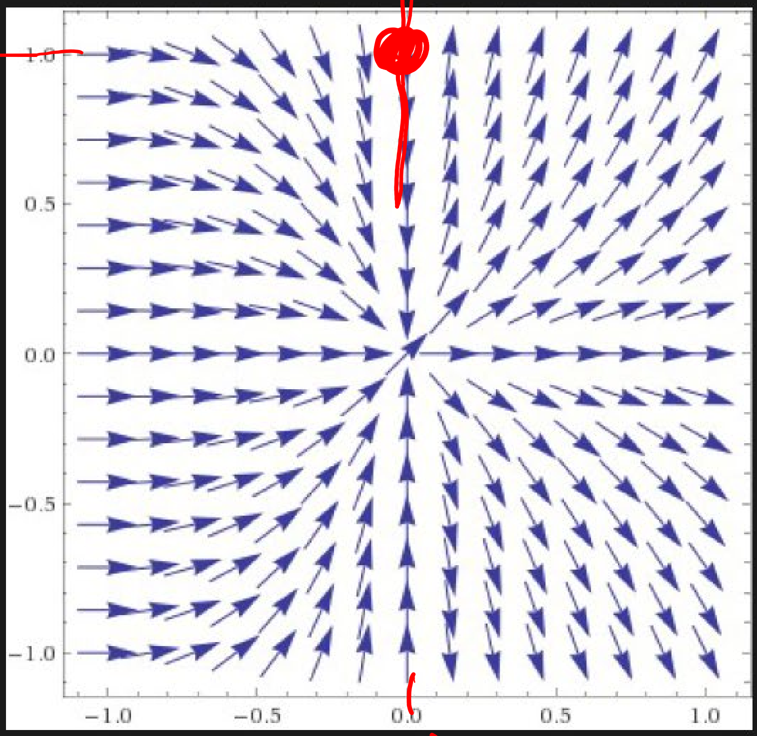
http://www.wolframalpha.com

slope field:

$\{1, y(1 + 1/x)\}/sqrt(1 + y \wedge 2(1 + 1/x) \wedge 2)$

sol'n should be functions

no soln



Ex 4 (IVP): $\cancel{dx}\,\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}}\,\cancel{dx}$, $y(0) = 5$. ← Calculus I

$$\int dy = \int \frac{1}{3}x^{-\frac{2}{3}}\,dx$$

General solution: $y = x^{\frac{1}{3}} + C$

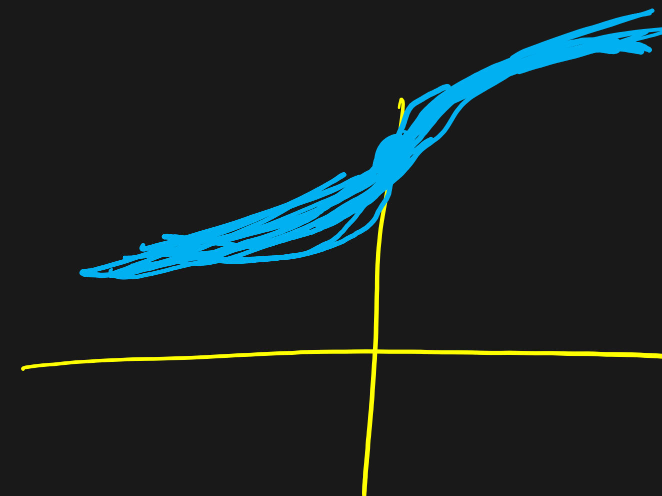
Initial value: $y(0) = 5$: $5 = 0^{\frac{1}{3}} + C$

Thus $C = 5$

$\boxed{\text{Thus IVP soln: } y = x^{\frac{1}{3}} + 5}$

$f(x) = x^{1/3} + 5$

$f : \mathbb{R} \to \mathbb{R}$



Section 2.4: Existence and Uniqueness.

In general, for IVP $y' = f(t, y),\ y(t_0) = y_0$ solution may or may not exist and solution may or may not be unique.

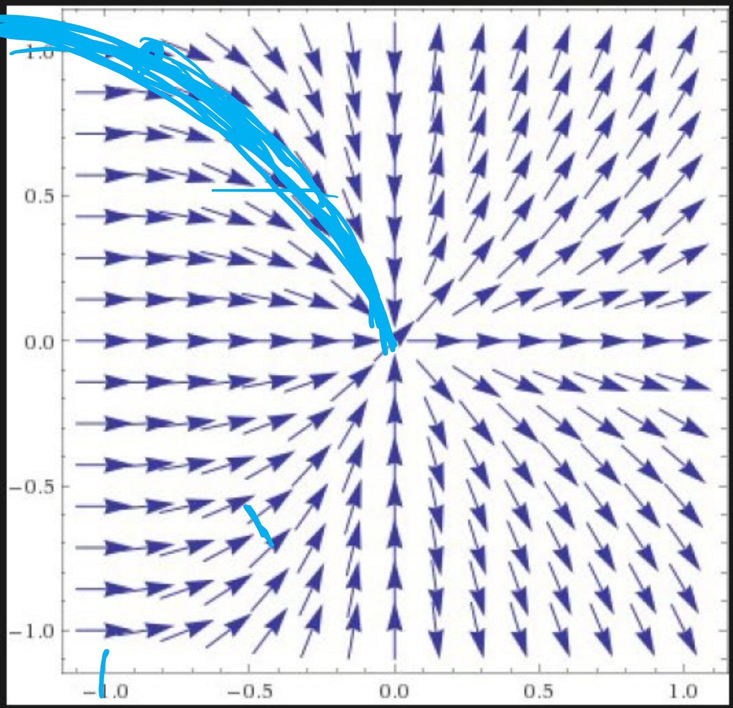
1.) Does there exist a solution to the DE $y' = f(t, y)$?

- If no, then solution to IVP $y' = f(t, y)$, $y(t_0) = y_0$ does not exist.
- If yes, then general solution to DE $y' = f(t, y)$ will include a constant term C.

2.) If solution exists to DE $y' = f(t, y)$, then does there exist a solution for C to IVP, $y(t_0) = y_0$ and is that solution unique.

Ex 3b (IVP): $\frac{dy}{dx} = y(1 + \frac{1}{x})$, $y(-1) = 1$

http://www.wolframalpha.com



$\boxed{\text{unique soln}}$

The general solution is $y = Cxe^x$

$y(-1) = 1$: $1 = C(-1)e^{-1}$ implies $C = -e$

IVP solution: $y = -exe^x = -xe^{x+1} = y$

Example Non-unique: $y' = y^{\frac{1}{3}}$, $\underline{y(3) = 0}$ ✩ ✩

$$y' = y^{1/3}$$

Solve: $\frac{dt}{y^{1/3}} \frac{dy}{\cancel{dt}} = \frac{\cancel{y^{1/3}}}{\cancel{y^{1/3}}} dt$

$$\int y^{-1/3}\, dy = \int dt$$

$$\left(y^{2/3} = \right)^{3/2} \quad \frac{3}{2}\left[\frac{3}{2} y^{2/3}\right] = \left[t + C\right]\frac{3}{2} = \left(\frac{3}{2}t + C\right)^{3/2}$$

Example Non-unique: $y' = y^{\frac{1}{3}}, \;\; y(3) = 0$

$$y = \pm\sqrt{\left(\tfrac{2}{3}(x) + C\right)^3}$$

IVP $\;\; y(3) = 0$:

$$0 = \pm\sqrt{\left(\tfrac{2}{3}(3) + c\right)^3}$$

$$0 = \pm\sqrt{(2 + c)^3}$$

$$0 = 2 + c \implies c = -2$$

Example Non-unique: $y' = y^{\frac{1}{3}}, \;\; y(3) = 0$

But $\quad y = \sqrt{(\frac{2}{3}x - 2)^3}$ and $y = -\sqrt{(\frac{2}{3}x - 2)^3}$

are not the only solutions to this IVP.

$$\boxed{y' = y^{1/3}}, \quad y(3) = 0$$

$$0 = 0^{1/3}$$

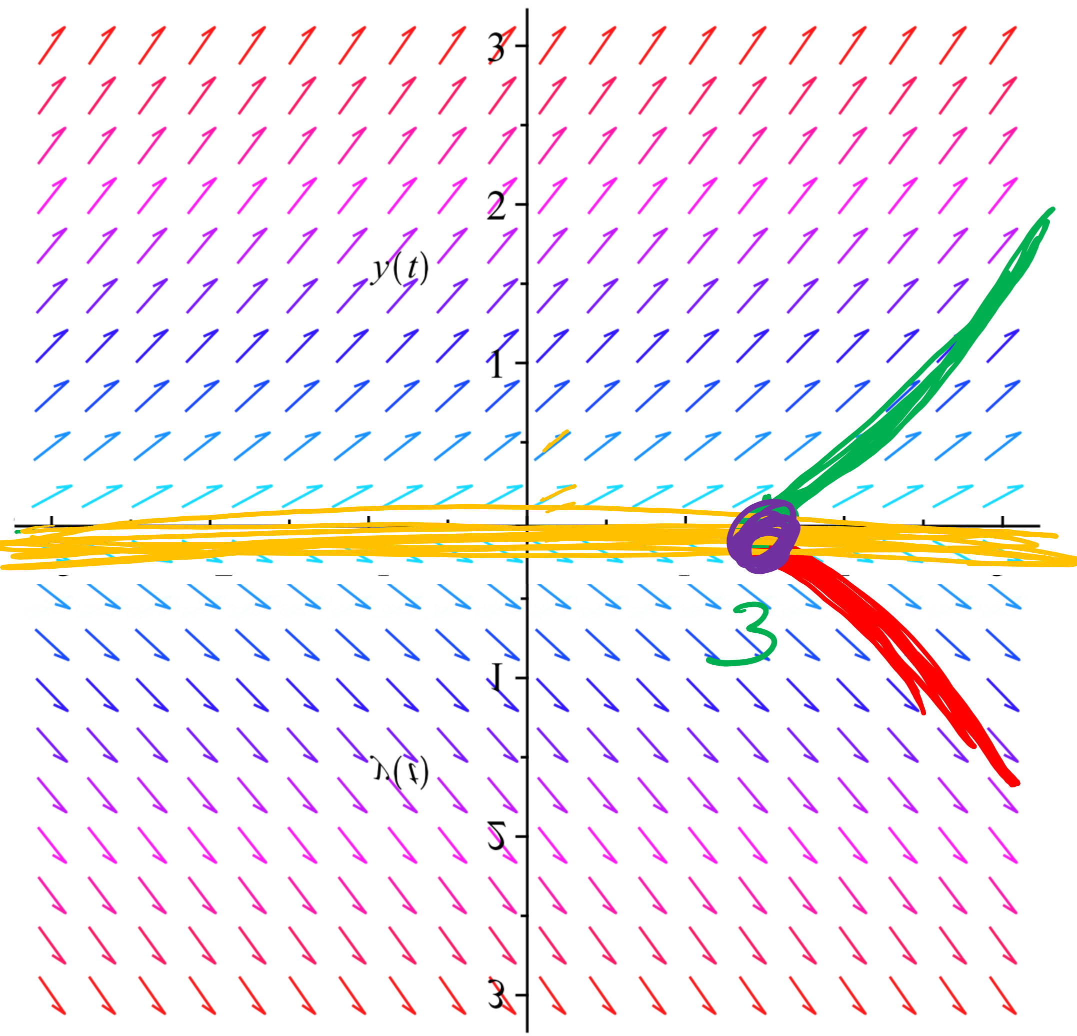
$$y = \left(\frac{2}{3}x - 2\right)^{3/2}$$

$$y = -\left(\frac{2}{3}x - 2\right)^{3/2}$$

$$y = 0 \Rightarrow y' = 0, \quad y(3) = 0$$

Classic counter-example slope field: $y' = y^{\frac{1}{3}}$

equil soln $y = 0$



$\boxed{y(0) = 3}$

Example Non-unique: $y' = y^{\frac{1}{3}}, \;\; y(3) = 0$

But $\quad y = \sqrt{(\frac{2}{3}x - 2)^3}$ and $y = -\sqrt{(\frac{2}{3}x - 2)^3}$

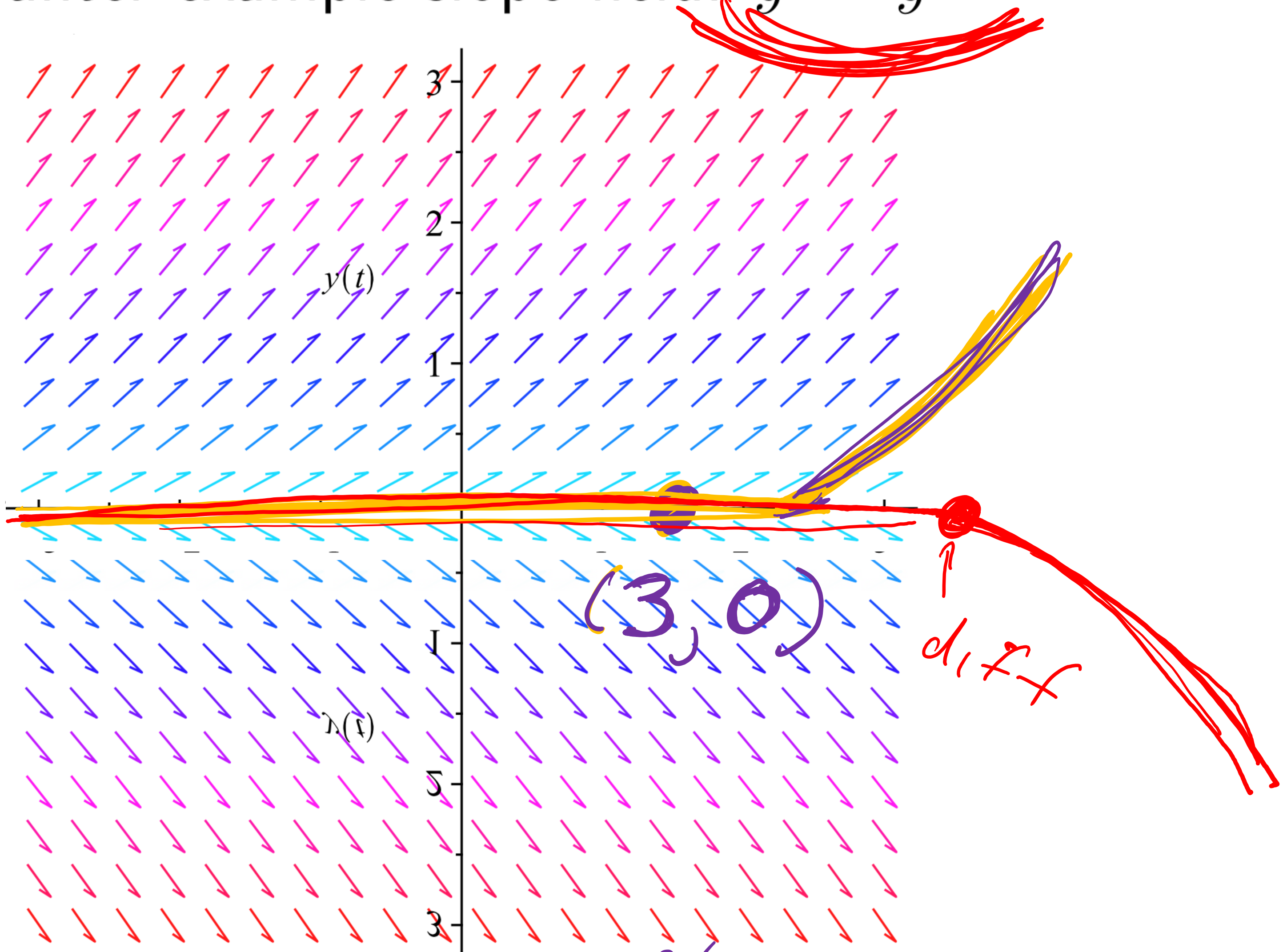
are not the only solutions to this IVP.

$y = 0$ or

$$y = \begin{cases} 0 & x \le A \\ \pm\sqrt{(\frac{2}{3}x - \frac{2}{3}A)} & x \ge A \end{cases}$$

where $A \geq 3$

Classic counter-example slope field: $y' = y^{\frac{1}{3}}$



$$y = 0 \longrightarrow y = \pm\left(\frac{2}{3}x + C\right)^{3/2}$$

Example Non-unique: $y' = y^{\frac{1}{3}}, \ \ y(3) = 0$

$y = 0$ is a solution to $y' = y^{\frac{1}{3}}$

since $y' = 0 = 0^{\frac{1}{3}} = y^{\frac{1}{3}}$

and $y(3) = 0$ since $y(t) = 0 \forall t$

Suppose $y \neq 0$. Then $\frac{dy}{dx} = y^{\frac{1}{3}}$ implies $y^{-\frac{1}{3}} dy = dx$

$\int y^{-\frac{1}{3}} dy = \int dx$ implies $\frac{3}{2} y^{\frac{2}{3}} = x + C$

$y^{\frac{2}{3}} = \frac{2}{3}x + C$ implies $y = \pm\sqrt{(\frac{2}{3}x + C)^3}$

General solution:

$$y = 0 \quad \text{or} \quad y = \begin{cases} 0 & t \leq -\frac{3}{2}C \\ \pm\sqrt{(\frac{2}{3}x + C)^3} & t \geq -\frac{3}{2}C \end{cases}$$

General solution:

$$y = 0 \quad \text{or} \quad y = \begin{cases} 0 & t \le -\frac{3}{2}C \\ \pm\sqrt{(\frac{2}{3}x + C)^3} & t \ge -\frac{3}{2}C \end{cases}$$

IVP: $y(3) = 0$. Then $0 = \sqrt{(2 + C)^3} \ \Rightarrow \ C = -2.$

The IVP, $y' = y^{\frac{1}{3}}$, $y(3) = 0$, has an infinite # of sol'ns:

$y = 0$, $\quad y = \sqrt{(\frac{2}{3}x - 2)^3}$, $\quad y = -\sqrt{(\frac{2}{3}x - 2)^3}$, and ...

General solution:

$$y = 0 \quad \text{or} \quad y = \begin{cases} 0 & t \le -\frac{3}{2}C \\ \pm\sqrt{(\frac{2}{3}x + C)^3} & t \ge -\frac{3}{2}C \end{cases}$$

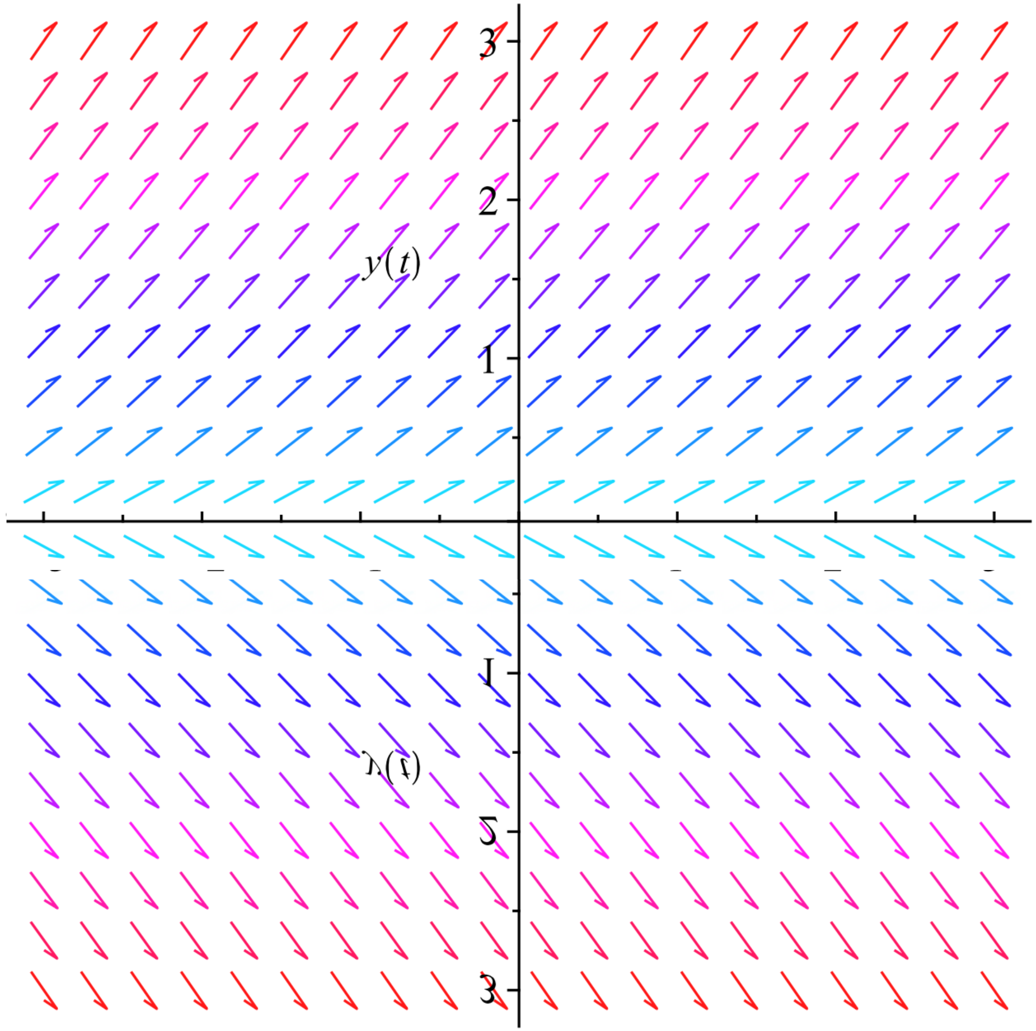
IVP: $\quad y(3) = 0$. Then $0 = \sqrt{(2 + C)^3} \;\Rightarrow\; C = -2.$

The IVP, $y' = y^{\frac{1}{3}}$, $y(3) = 0$, has an infinite # of sol'ns:

$$y = 0 \quad \text{or} \quad y = \begin{cases} 0 & t \le A \\ \pm\sqrt{(\frac{2}{3}x - \frac{2}{3}A)^3} & t \ge A \end{cases} \quad \text{where } A \ge 3$$

IVP soln

Classic counter-example slope field: $y' = y^{\frac{1}{3}}$



Special cases:

When do we know a unique solution exists?

Friday

Breakout groups: work on problem 2.3:18a. Use short-cut: $$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v$$

N 16. A ball with mass 0.15 kg is thrown upward with initial velocity 20 m/s from the roof of a building 30 m high. Neglect air resistance.

a. Find the maximum height above the ground that the ball reaches.

N 17. Assume that the conditions are as in Problem 16 except that there is a force due to air resistance of magnitude $|v|/30$ directed opposite to the velocity, where the velocity v is measured in m/s.

a. Find the maximum height above the ground that the ball reaches.

v & t ⟹ v & x

N 18. Assume that the conditions are as in Problem 16 except that there is a force due to air resistance of magnitude $v^2/1325$ directed opposite to the velocity, where the velocity v is measured in m/s.

a. Find the maximum height above the ground that the ball reaches.