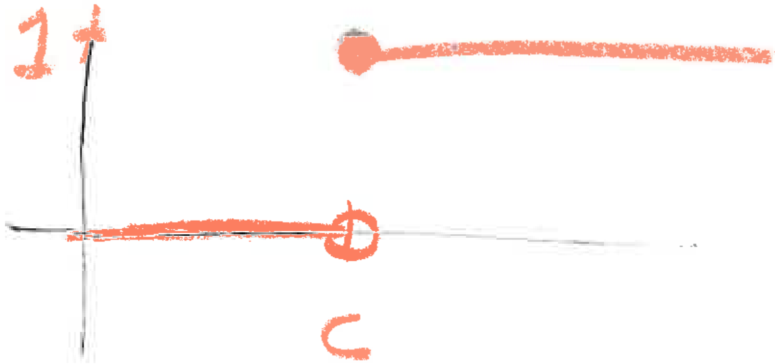
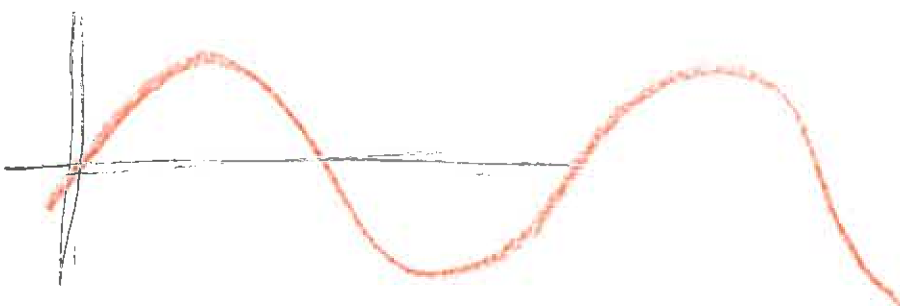
6.3: Step functions.

Graph $u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$



Graph $g(t) = sin(t)$.

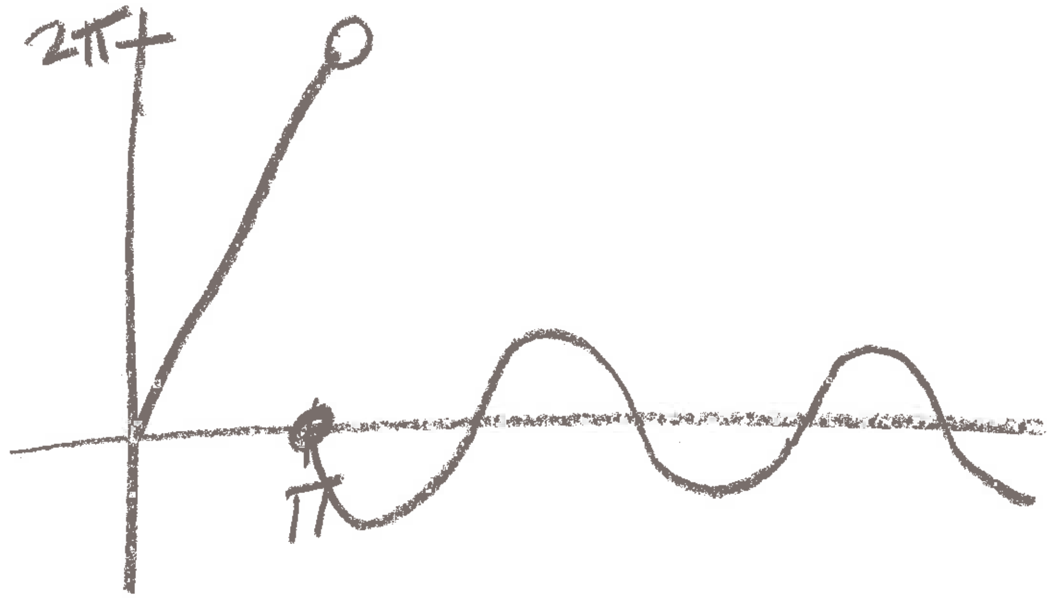


Graph $h(t) = u_{\pi}(t)sin(t)$.

Graph $f(t) = 2t + \overset{\text{0 or 1}}{u_\pi(t)}[sin(t) - 2t] = \begin{cases} 2t & t < \pi \\ sin(t) & t \geq \pi \end{cases}$

$t < \pi$, $f(t) = 2t + 0 = 2t$

$t \geq \pi$, $f(t) = \cancel{2t} + (1)[sin(t) - \cancel{2t}]$



$$h(t) = \begin{cases} t & 0 \leq t < 4 \\ ln(t) & t \geq 4 \end{cases}$$

implies $h(t) = t + u_4(t)[ln(t) - t]$

✓ Check:

$2 = h(2) = 2 + \overset{0}{\cancel{u_4(2)}}[ln(2) - 2] = 2$ ✓

$ln(5) = h(5) = 5 + \underbrace{u_4(5)}_{\text{"1"}}[ln(5) - 5] = \cancel{5} + ln(5) - \cancel{5}$

$= ln(5)$ ✓

Example: $f(t) = \begin{cases} f_1, & \text{if } t < 4; \\ f_2, & \text{if } 4 \le t < 5; \\ f_3, & \text{if } 5 \le t < 10; \\ f_4, & \text{if } t \ge 10; \end{cases}$

Hence

$$\boxed{\begin{aligned} f(t) = f_1(t) + u_4(t)[f_2(t) - f_1(t)] + u_5(t)[f_3(t) - f_2(t)] \\ +u_{10}(t)[f_4(t) - f_3(t)] \end{aligned}}$$

Partial check:

If $t = 3$: $f(3) = f_1(3) + 0[f_2(3) - f_1(3)]$

$$+0[f_3(3) - f_2(3)] + 0[f_4(3) - f_3(3)] = f_1(3)$$

If $t = 9$: $f(9) = f_1(9) + 1[f_2(9) - f_1(9)]$

$$+1[f_3(9) - f_2(9)] + 0[f_4(9) - f_3(9)] = f_3(9)$$

Examples:

$f(t) = \begin{cases} 0 & 0 \le t < 2 \\ t^2 & t \ge 2 \end{cases}$ implies $f(t) = 0 + u_2(t)[t^2 - 0]$

$= u_2(t) \cdot t^2$

$g(t) = \begin{cases} t^2 & 0 \le t < 3 \\ 0 & t \ge 3 \end{cases}$ implies $g(t) = t^2 + u_3(t)[0 - t^2]$

$= t^2$

$= t^2 - u_3(t) t^2$

$$j(t) = \begin{cases} t & 0 \le t < 5 \\ 2 & 5 \le t < 8 \\ e^t & t \ge 8 \end{cases} \quad \text{implies}$$

$$j(t) = t + u_5(t)[2-t] + u_8(t)[e^t - 2]$$

Formula 13: $\mathcal{L}(u_c(t)f(t-c)) = e^{-cs}F(s) = e^{-cs}\mathcal{L}(f(t))$

Let $\boxed{g(t) = f(t+c)}$

$$g(t-c) = f(t-c+c) = f(t)$$

$$\mathcal{L}(u_c(t)\cdot f(t)) = \mathcal{L}(u_c(t)\cdot g(t-c))$$

$$= e^{-cs}G(s) \quad \text{by formula 13}$$

formula 13

$$= e^{-cs}\mathcal{L}(g(t))$$

$$= e^{-cs}\mathcal{L}(f(t+c))$$

Formula 13'

$$\mathcal{L}(u_c(t)\cdot f(t)) = e^{-cs}\mathcal{L}(f(t+c))$$

Formula 13: $\boxed{\mathcal{L}(u_c(t)f(\underbrace{t-c}_{\text{Let } x=t-c \Rightarrow t=x+c}))=e^{-cs}\mathcal{L}(f(t))}$.

substitution applied only to f, not $u_c(t)$

Let $g(t)=f(t+c)$. Then $g(t-c)=f(t-c+c)=f(t)$. Thus

$$\mathcal{L}(u_c(t)f(t))=\mathcal{L}(u_c(t)g(t-c))=e^{-cs}\mathcal{L}(g(t))$$
$$=e^{-cs}\mathcal{L}(f(t+c)).$$

t-world → s-word

Formula 13' or equivalently

$$\boxed{\mathcal{L}(u_c(t)f(t))=e^{-cs}\mathcal{L}(f(t+c)).}$$

In other words, replacing $t-c$ with t is equivalent to replacing t with $t+c$

Find the LaPlace transform of the following:

a.) $\mathcal{L}(u_3(t)(t^2-2t+1))=$ ____________________

since taking LaPlace transform $(t \to s)$

use 13' : $\mathcal{L}(u_c(t)f(t))=e^{-cs}\mathcal{L}(f(t+c))$

$\mathcal{L}(u_3(t)(t^2-2t+1))=e^{-3s}\mathcal{L}(f(t+3))$

where $f(t)=t^2-2t+1$

$\Rightarrow f(t+3)=(t+3)^2-2(t+3)+1$

$=t^2+6t+9-2t-6+1=t^2+4t+4$

$$\mathcal{L}(u_3(t)\cdot(t^2-2t+1))$$

$$= e^{-3s}\,\mathcal{L}(f(t+3))$$

$$= e^{-3s}\,\mathcal{L}((t+3)^2-2(t+3)+1))$$

$$= e^{-3s}\left[\mathcal{L}(t^2+4t+4)\right]$$

$$= e^{-3s}\left[\mathcal{L}(t^2)+4\mathcal{L}(t)+4\mathcal{L}(1)\right]$$

$$= e^{-3s}\left[\frac{2!}{s^3}+\frac{4}{s^2}+\frac{4}{s}\right]$$

$$\mathcal{L}(t^n)=\frac{n!}{s^{n+1}}$$

$$= e^{-3s}\left[\frac{2}{s^3}+\frac{4}{s^2}+\frac{4}{s}\right]$$

b.) $\mathcal{L}(u_4(t)(e^{-8t})) =$ last week

c.) $\mathcal{L}(u_2(t)(t^2e^{3t})) =$ See answers

Find the LaPlace transform of

d.) $g(t) = \begin{cases} 0 & t < 3 \\ e^{t-3} & t \geq 3 \end{cases}$ See answers

e.) $f(t) = \begin{cases} 0 & t < 3 \\ 5 & 3 \leq t < 4 \\ t-5 & t \geq 4 \end{cases}$

$f(t) = 0 + u_3(5-0) + u_4(t-5$

$= 5u_3(t) + u_4(t)(t-10$

Formula 13: $\mathcal{L}(u_c(t)f(t-c)) = e^{-cs}\mathcal{L}(f(t))$.

Let $F(s) = \mathcal{L}(f(t))$.

Then $\mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}(\mathcal{L}(f(t))) = f(t)$.

Thus $\mathcal{L}^{-1}(e^{-cs}F(s))$

$$= \mathcal{L}^{-1}(e^{-cs}\mathcal{L}(f(t))) = u_c(t)f(t-c)$$

where $f(t) = \mathcal{L}^{-1}(F(s))$

Find the inverse LaPlace transform of the following:

a.) $\mathcal{L}^{-1}(e^{-8s}\frac{1}{s-3}) =$ __________

b.) $\mathcal{L}(u_4(t)(e^{-8t})) =$ Last week

c.) $\mathcal{L}(u_2(t)(t^2e^{3t})) =$ $e^{-2s}\mathcal{L}((t+2)^2 e^{3(t+2)})$

$e^{-2s}\mathcal{L}((t^2+4t+4)e^{3t}e^{6})$

constant

Find the LaPlace transform of

$= e^{-2s}e^{6}\left[\mathcal{L}(t^2e^{3t}) + 4\mathcal{L}(te^{3t}) + 4\mathcal{L}(e^{3t}\cdot 1)\right]$

d.) $g(t) = \begin{cases} 0 & t < 3 \\ e^{t-3} & t \geq 3 \end{cases}$

e.) $f(t) = \begin{cases} 0 & t < 3 \\ 5 & 3 \leq t < 4 \\ t-5 & t \geq 4 \end{cases}$

Formula 13: $\mathcal{L}(u_c(t)f(t-c)) = e^{-cs}\mathcal{L}(f(t))$.

Let $F(s) = \mathcal{L}(f(t))$. $= e^{-2s+6}\left[\frac{2}{(s-3)^3} + \frac{4}{(s-3)^2} + \frac{4}{s-3}\right]$

Then $\mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}(\mathcal{L}(f(t))) = f(t)$.

Thus $\mathcal{L}^{-1}(e^{-cs}F(s))$ $\boxed{\mathcal{L}(e^{as}f(t)) \Rightarrow s \to s-a \text{ in } \mathcal{L}(f))}$

$$= \mathcal{L}^{-1}(e^{-cs}\mathcal{L}(f(t))) = u_c(t)f(t-c)$$

where $f(t) = \mathcal{L}^{-1}(F(s))$

Find the inverse LaPlace transform of the following:

a.) $\mathcal{L}^{-1}(e^{-8s}\frac{1}{s-3}) =$ ____________