If $y(0) = 4$, then $4 = Ce^{3(0)}$ implies $C = 4$.

Thus by existence and uniqueness thm, $y = 4e^{3t}$ is the unique solution to IVP: $2y' + 6y = 0$, $y(0) = 4$.

CH 2: Solve $\frac{dy}{dt} = f(t, y)$ for special cases:

2.2: Separation of variables: $N(y)dy = P(t)dt$

2.1: First order linear eqn: $1\frac{dy}{dt} + \boxed{p(t)}y = g(t)$

Ex 1: $t^2y' + 2ty = tsin(t)$

Ex 2: $y' = ay + b$

Ex 3: $y' + 3t^2y = t^2$, $y(0) = 0$

use an integrating factor $u(t) = e^{\int p(t)dt}$

Note: can use either section 2.1 method (integrating factor) or 2.2 method (separation of variables) to solve ex 2 and 3.

Ex 1: $t^2y' + 2ty = sin(t)$
(note, cannot use separation of variables).

$t^2y' + 2ty = sin(t)$

$\int (t^2y)' = \int sin(t)$ implies $\int (t^2y)'dt = \int sin(t)dt$

$(t^2y) = -cos(t) + C$ implies $y = -t^{-2}cos(t) + Ct^{-2}$

PRODUCT RULE

Ex. 2: Solve $\frac{dy}{dt} = ay + b$ by separating variables:

$\frac{dy}{ay+b} = dt \Rightarrow \int \frac{dy}{ay+b} = \int dt \Rightarrow \frac{ln|ay+b|}{a} = t + C$

$ln|ay + b| = at + C$ implies $e^{ln|ay+b|} = e^{at+C}$

$|ay + b| = e^C e^{at}$ implies $ay + b = \pm(e^C e^{at})$

$ay = Ce^{at} - b$ implies $y = Ce^{at} - \frac{b}{a}$

Gen ex: Solve $1y' + \boxed{p(x)}y = g(x)$ $\qquad \int p(x)dx = F(x)$

Let $F(x)$ be an anti-derivative of $p(x)$. Thus $p(x) = F'(x)$

Better notation

$e^{F(x)}y' + [p(x)e^{F(x)}]y = g(x)e^{F(x)}$

$e^{F(x)}y' + [F'(x)e^{F(x)}]y = g(x)e^{F(x)}$ $\qquad \int_{x_0}^{x} p(s)ds = F(x)$

$\int [e^{F(x)}y]' = \int g(x)e^{F(x)}$

$e^{F(x)}y = \int g(x)e^{F(x)}dx$

$y = e^{-F(x)} \int g(x)e^{F(x)}dx$

Integrating Factor $e^{\int p(x)dx}$

integrating factor

If $y(0) = 4$, then $4 = Ce^{3(0)}$ implies $C = 4$.

Thus by existence and uniqueness thm, $y = 4e^{3t}$ is the unique solution to IVP: $2y' + 6y = 0$, $y(0) = 4$.

CH 2: Solve $\frac{dy}{dt} = f(t, y)$ for special cases:

2.2: Separation of variables: $N(y)dy = P(t)dt$

2.1: First order linear eqn: $\frac{dy}{dt} + p(t)y = g(t)$

Ex 1: $t^2y' + 2ty = tsin(t)$

Ex 2: $y' = ay + b$

Ex 3: $y' + 3t^2y = t^2$, $y(0) = 0$

Note: can use either section 2.1 method (integrating factor) or 2.2 method (separation of variables) to solve ex 2 and 3.

Ex 1: $t^2y' + 2ty = sin(t)$
(note, cannot use separation of variables).

product rule

$t^2y' + 2ty = sin(t)$

$(t^2y)' = \int sin(t)$ implies $\int (t^2y)'dt = \int sin(t)dt$

$(t^2y) = -cos(t) + C$ implies $y = -t^{-2}cos(t) + Ct^{-2}$

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product rule

Ex. 2: Solve $\frac{dy}{dt} = ay + b$ by separating variables:

$\frac{dy}{ay+b} = dt \;\Rightarrow\; \int \frac{dy}{ay+b} = \int dt \;\Rightarrow\; \frac{ln|ay+b|}{a} = t + C$

$ln|ay + b| = at + C$ implies $e^{ln|ay+b|} = e^{at+C}$

$|ay + b| = e^Ce^{at}$ implies $ay + b = \pm(e^Ce^{at})$

$ay = Ce^{at} - b$ implies $y = Ce^{at} - \frac{b}{a}$

Gen ex: Solve $1y' + p(x)y = g(x)$

Let $F(x)$ be an anti-derivative of $p(x)$. Thus $p(x) = F'(x)$

$e^{F(x)}y' + [p(x)e^{F(x)}]y = g(x)e^{F(x)}$

$e^{F(x)}y' + [F'(x)e^{F(x)}]y = g(x)e^{F(x)}$

$[e^{F(x)}y]' = \int g(x)e^{F(x)}$

$e^{F(x)}y = \int g(x)e^{F(x)}dx$

$y = e^{-F(x)} \int g(x)e^{F(x)}dx$

integrating factor $u = e^{\int p(x)dx} = e^{F(x)}$

Integration by parts:

Derivative of a product: $(uv)' = uv' + vu'$

$$uv' = (uv)' - vu'$$

$$\int uv' = \int (uv)' - \int vu'$$

$$\boxed{\int uv' = (uv) - \int vu'}$$

Example: $\int e^{2x} sin(3x)$

Let $u = sin(3x)$, $dv = e^{2x}$

then $du = 3cos(3x)$, $v = \frac{1}{2}e^{2x}$

then $d^2u = -9sin(3x)$, $\int v = \frac{1}{4}e^{2x}$

$$\int e^{2x} sin(3x) = \frac{1}{2} sin(3x)e^{2x} - \int \frac{3}{2} e^{2x} cos(3x)$$

$$= \frac{1}{2} sin(3x)e^{2x} - [\frac{3}{4} cos(3x)e^{2x} - \int \frac{-9}{4} sin(3x)e^{2x}$$

$$\int e^{2x} sin(3x) = \frac{1}{2} sin(3x)e^{2x} - \frac{3}{4} cos(3x)e^{2x} - \frac{9}{4} \int sin(3x)e^{2x}$$

$$\frac{13}{4} \int e^{2x} sin(3x) = \frac{1}{2} sin(3x)e^{2x} - \frac{3}{4} cos(3x)e^{2x}$$

$$\int e^{2x} sin(3x) = \frac{4}{13} [\frac{1}{2} sin(3x)e^{2x} - \frac{3}{4} cos(3x)e^{2x}]$$

Optional Exercise: Calculate $\int e^x cos(2x)$