If $\phi_0(t) = 0$ & $\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))\,ds$

hyp

Determine formula for ϕ_n:

Note patterns:

$\int_0^t s\,ds = \frac{t^2}{2} =$

$\int_0^t \frac{s^2}{2}ds = \frac{t^3}{3 \cdot 2} =$

$\int_0^t \frac{s^3}{3 \cdot 2}ds = \frac{t^4}{4 \cdot 3 \cdot 2} =$

$\int_0^t \frac{s^4}{4 \cdot 3 \cdot 2}ds = \frac{t^5}{5 \cdot 4 \cdot 3 \cdot 2} =$

Thus look for factorials.

$\phi_0(t) = 0$

$\phi_1(t) = \frac{t^2}{2}$

$\phi_2(t) = \frac{t^2}{2} + \frac{t^3}{3}$

$\phi_3(t) = \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6}$

$\phi_4(t) = \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6} + \frac{t^5}{15} = \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{3 \cdot 2} + \frac{t^5}{5 \cdot 3}$

Thus $\phi_n(t) =$

FYI (ie not on quizzes/exam): $f(t,y) = t + 2y$

Defn: $\sum_{k=0}^{\infty} a_k x^k = \lim_{n \to \infty} \sum_{k=0}^{n} a_k x^k$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots.$$

Taylor's Theorem: If f is analytic at 0, then for small x (i.e., x near 0),

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

$$= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \ldots$$

Example:

$e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!}$ and thus $e^{bt} = \sum_{k=0}^{\infty} \frac{b^k t^k}{k!}$ for t near 0.

conc

$\Downarrow$

$$\phi_n(t) = \sum_{k=2}^{n} \frac{2^{k-2}}{k!} t^k$$

Thus $\phi(t) = \lim_{n \to \infty} \phi_n(t) = \sum_{k=2}^{\infty} \frac{2^{k-2}}{k!} t^k = \frac{1}{4} \sum_{k=2}^{\infty} \frac{2^k}{k!} t^k$

$$= \frac{1}{4}(\quad - \quad - \quad)$$

Construct ϕ using method of successive approximation – also called Picard's iteration method.

Let $\boxed{\phi_0(t) = 0}$ (or the function of your choice)

Let $\phi_1(t) = \int_0^t f(s, \phi_0(s))ds$

Let $\phi_2(t) = \int_0^t f(s, \phi_1(s))ds$

$\vdots$

Let $\phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))ds$

Let $\phi(t) = \lim_{n\to\infty} \phi_n(t)$

To finish the proof, need to answer the following questions (see book or more advanced class):

1.) Does $\phi_n(t)$ exist for all n?

2.) Does sequence ϕ_n converge?

3.) Is $\phi(t) = \lim_{n\to\infty} \phi_n(t)$ a solution to (*).

4.) Is the solution unique.

Example: $\boxed{y' = t + 2y. \quad \text{That is } f(t, y) = t + 2y}$

Let $\phi_0(t) = 0$

Let $\phi_1(t) = \int_0^t f(s, 0)ds = \int_0^t (s + 2(0))ds$

$$= \int_0^t s\,ds = \frac{s^2}{2}\Big|_0^t = \frac{t^2}{2}$$

Let $\phi_2(t) = \int_0^t f(s, \phi_1(s))ds = \int_0^t f(s, \frac{s^2}{2})ds$

$$= \int_0^t (s + 2(\tfrac{s^2}{2}))ds = \frac{t^2}{2} + \frac{t^3}{3}$$

Let $\phi_3(t) = \int_0^t f(s, \phi_2(s))ds = \int_0^t f(s, \frac{s^2}{2} + \frac{s^3}{3})ds$

$$= \int_0^t (s + 2(\tfrac{s^2}{2} + \tfrac{s^3}{3}))ds = \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6}$$

Let $\phi_4(t) = \int_0^t f(s, \phi_3(s))ds$

$$= \int_0^t f(s, \tfrac{s^2}{2} + \tfrac{s^3}{3} + \tfrac{s^4}{6})ds$$

$$= \int_0^t (s + 2(\tfrac{s^2}{2} + \tfrac{s^3}{3} + \tfrac{s^4}{6}))ds$$

$$= \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{6} + \frac{t^5}{15}$$

$\vdots$