Recall that a constant solution is an equilibrium solution. Thus its derivative is 0.

To find an equilibrium solution (i.e., constant solution), plug it in (for example, plug in $y(t) = k$ or $x_1(t) = k_1, x_2(t) = k_2$ depending on variables used and if you have one DE or a system of two DEs). Since the derivative of a constant is zero, this is equivalent to setting the derivative $= 0$.

Find all equilibrium solutions and classify them (stable, asymptotically stable, semi-stable, unstable and if system of DEs, node, saddle, spiral, center). In the case of non-linear system of DEs, state all possibilities for type of equilibrium solution.

If the (system of) differential equation(s) does not have an equilibrium solution, state so (note 4 of the following 16 problems below do not have an equilibrium solution).

Hint: The eigenvalues of upper and lower triangular matrices are the diagonal entries.
Note: You do not need to draw any direction fields.

1.) $y' = (y-3)^4(y-5)^9$ $\quad y = 3$ is semi-stable, $y = 5$ is unstable.

2.) $y' = y^2 + 2$ $\quad$ no equilibrium solution.

3.) $y' = sin(y)$ $\quad y = 2n\pi$ is unstable, $y = (2n+1)\pi$ is asymptotically stable.

4.) $y' = sin(t)$ $\quad$ no equilibrium solution.

5.) $y' = sin^2(y)$ $\quad y = n\pi$ is semi-stable.

6.) $y' = sin^2(t)$ $\quad$ no equilibrium solution.

7.) ~~$y' = ty$ $\quad y = 0$ is unstable.~~

8.) $x' = 4 - y^2, \;\; y' = (x+1)(y-x)$

If $4 - y^2 = 0$, then $y = \pm 2$

If $y = 2$, then $(x+1)(y-x) = (x+1)(2-x) = 0$. Thus $x = -1, 2$.

If $y = -2$, then $(x+1)(y-x) = (x+1)(-2-x) = 0$. Thus $x = -1, -2$.

Jacobian matrix: $\begin{bmatrix} 0 & -2y \\ y-2x-1 & x+1 \end{bmatrix}$

For $(x, y) = (-1, 2)$, Jacobian matrix is $\begin{bmatrix} 0 & -4 \\ 3 & 0 \end{bmatrix}$

$r = \dfrac{0 \pm \sqrt{0-4(12)}}{2} = 0 \pm bi$

$\varepsilon > 0$

$-\varepsilon \pm bi$

$\varepsilon \pm bi$

$\varepsilon > 0$

Thus $(x(t), y(t)) = (-1, 2)$ is a stable center or unstable spiral or asymptotically stable spiral.

For $(x, y) = (2, 2)$, Jacobian matrix is $\begin{bmatrix} 0 & -4 \\ -3 & 3 \end{bmatrix}$

Thus $(x(t), y(t)) = (2, 2)$ is an unstable saddle.

$r_1 > 0$

$r_2 < 0$

$r = \dfrac{3 \pm \sqrt{9-4(-12)}}{2}$

For $(x, y) = (-1, -2)$, Jacobian matrix is $\begin{bmatrix} 0 & 4 \\ -1 & 0 \end{bmatrix}$

Thus $(x(t), y(t)) = (-1, -2)$ is a stable center or unstable spiral or asymptotically stable spiral.

For $(x, y) = (-2, -2)$, Jacobian matrix is $\begin{bmatrix} 0 & 4 \\ 1 & -1 \end{bmatrix}$

Thus $(x(t), y(t)) = (-2, -2)$ is an unstable saddle.

9.) $x' = x - 2, \;\; y' = x - 1$ no equilibrium solution.

10.) $\mathbf{x}' = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{x}$

One positive (1) and one negative eigenvalue (-2). Thus $(x_1(t), x_2(t)) = (0, 0)$ is an unstable saddle.

11.) $\mathbf{x}' = \begin{bmatrix} 1 & 0 \\ 5 & -2 \end{bmatrix} \mathbf{x}$

One positive (1) and one negative eigenvalue (-2). Thus $(x_1(t), x_2(t)) = (0, 0)$ is an unstable saddle.

12.) $\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{x}$

Purely imaginary eigenvalues $i, -i$. Thus $(x_1(t), x_2(t)) = (0, 0)$ is a stable center.

13.) $\mathbf{x}' = \begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix} \mathbf{x}$

Two positive eigenvalues 1, 2. Thus $(x_1(t), x_2(t)) = (0, 0)$ is an unstable node.

14.) $\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \mathbf{x}$

Two complex eigenvalues, $-1 \pm 2i$, with negative real part. Thus $(x_1(t), x_2(t)) = (0, 0)$ is an asymptotically stable spiral.

15.) $\mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -5 & 2 \end{bmatrix} \mathbf{x}$

Two complex eigenvalues, $1 \pm 2i$, with positive real part. Thus $(x_1(t), x_2(t)) = (0, 0)$ is an unstable spiral.

16.) $\mathbf{x}' = \begin{bmatrix} -1 & 0 \\ 5 & -2 \end{bmatrix} \mathbf{x}$

Two negative eigenvalues -1, -2. Thus $(x_1(t), x_2(t)) = (0, 0)$ is an asymptotically stable node.

Suppose an object moves in the 2D plane (the x_1, x_2 plane) so that it is at the point $(x_1(t), x_2(t))$ at time t. Suppose the object's velocity is given by

$$x_1'(t) = ax_1 + bx_2,$$
$$x_2'(t) = cx_1 + dx_2$$

Or in matrix form $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

To solve, find eigenvalues and corresponding eigenvectors:

$$\begin{vmatrix} a-r & b \\ c & d-r \end{vmatrix} = (a-r)(d-r) - bc = r^2 - (a+d)r + ad - bc = 0.$$

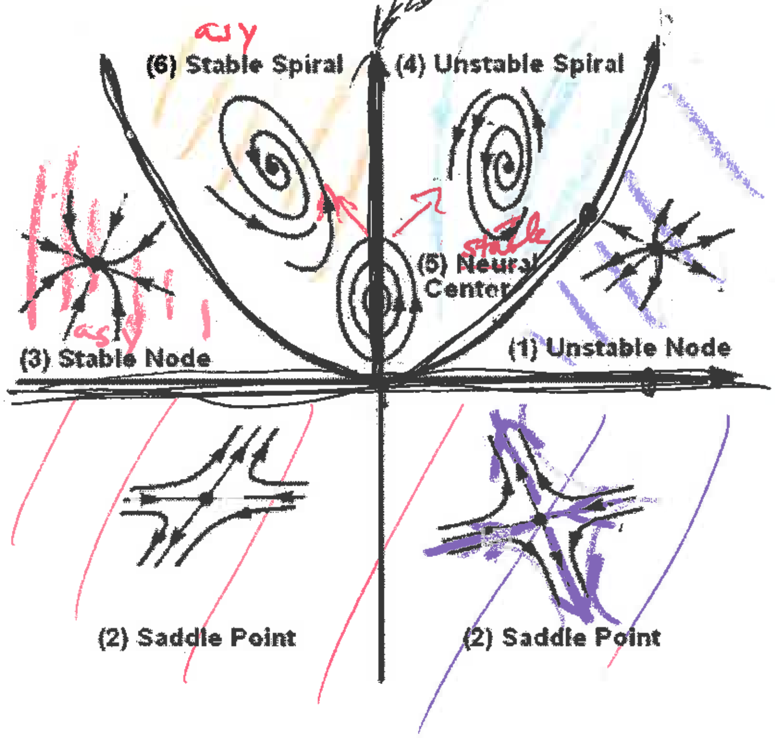
$$\text{Thus } r = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

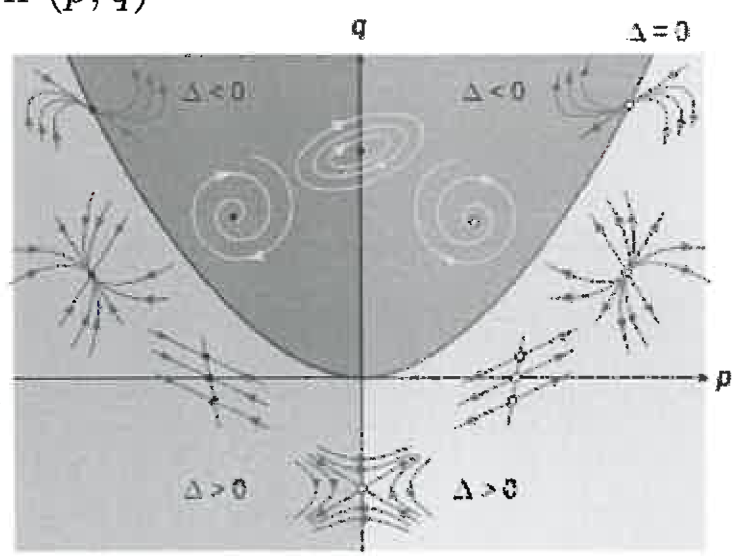
Let $p = trace(A) = a + d$ and let $q = detA = ad - bc$

stable cnts

Then $r = \frac{p \pm \sqrt{p^2 - 4q}}{2}$

Thus the type of solution depends on (p, q)





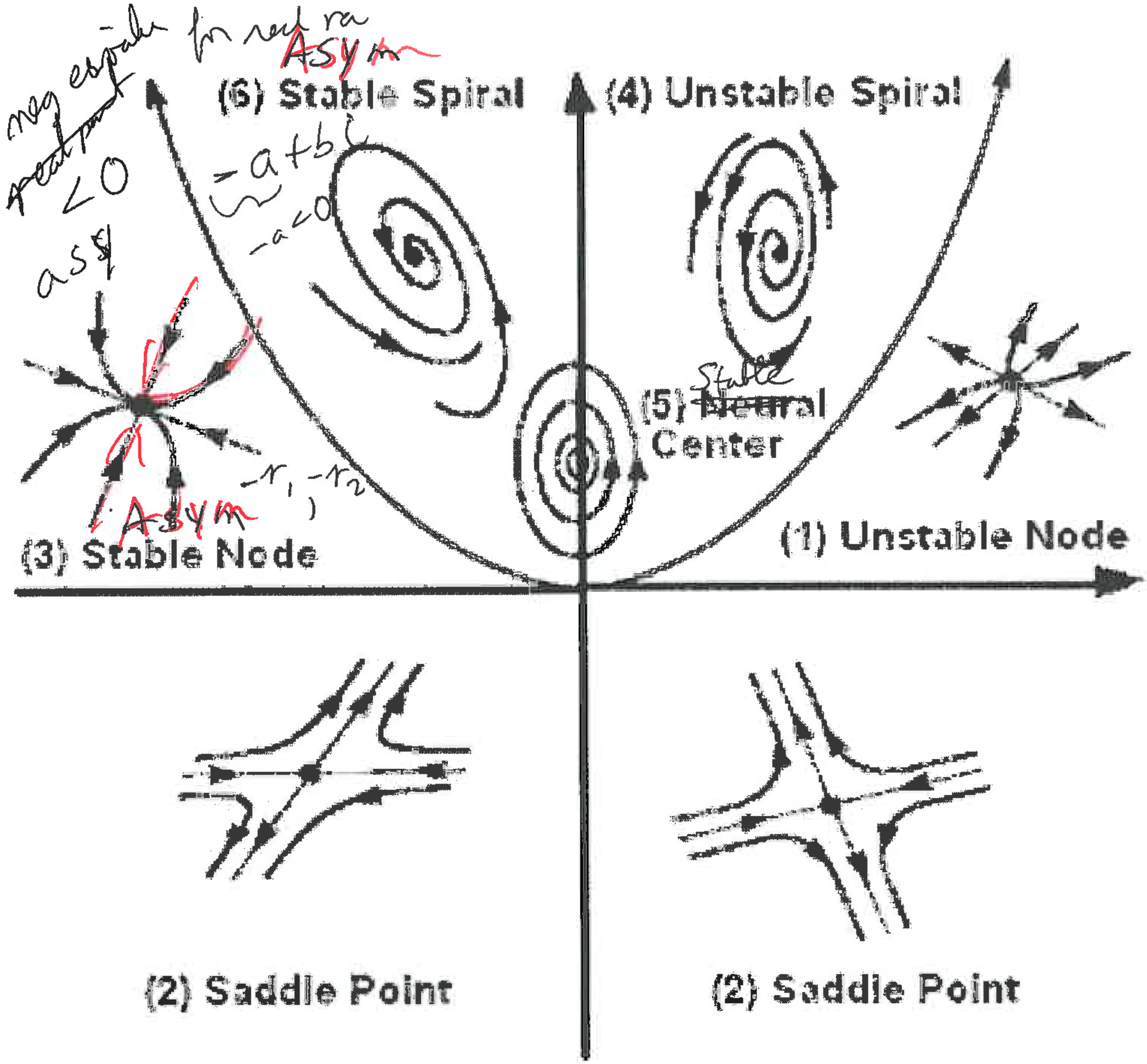
$\frac{dx}{dt} = Ax + By$ $\quad p = A + D$

$\frac{dy}{dt} = Cx + Dy$ $\quad q = AD - BC$

$\Delta = p^2 - 4q$

http://digital-library.theiet.org/content/journals/10.1049/iet-gtd_20080456

https://en.wikipedia.org/wiki/Phase_plane



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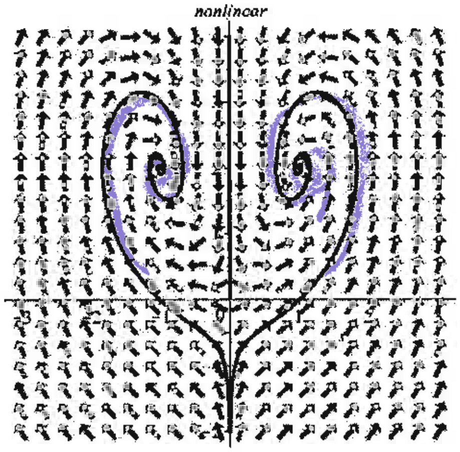
Problems 21-23 show the stream plot in the $x_1 - x_2$-plane for a system of two first order differential equations In addition to determining and classifying all equilibrium solutions, also draw the trajectories satisfying the initial values $(x_1(0), x_2(0)) = (0,1)$, $(x_1(0), x_2(0)) = (1,0)$, $(x_1(0), x_2(0)) = (1,2)$, $(x_1(0), x_2(0)) = (-1,0)$. Also describe the basins of attraction.

21.)



$(x_1(t), x_2(t)) = (0,0)$ is an unstable saddle.

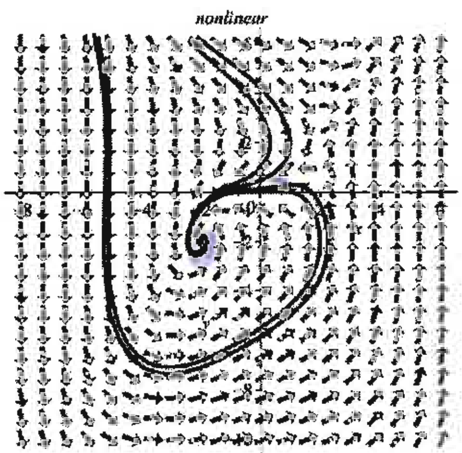
$(x_1(t), x_2(t)) = (1,2)$ is an asymptotically stable ~~node.~~ spiral

basin of attraction: $x_1 > 0$.

$(x_1(t), x_2(t)) = (-1,2)$ is an asymptotically stable ~~node~~. spirals

basin of attraction: $x_1 < 0$.

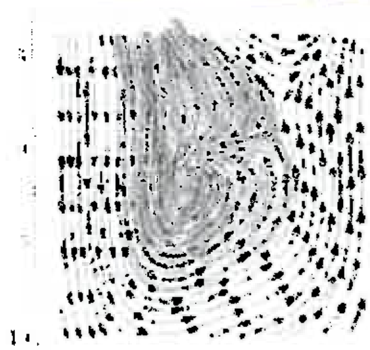
22.)



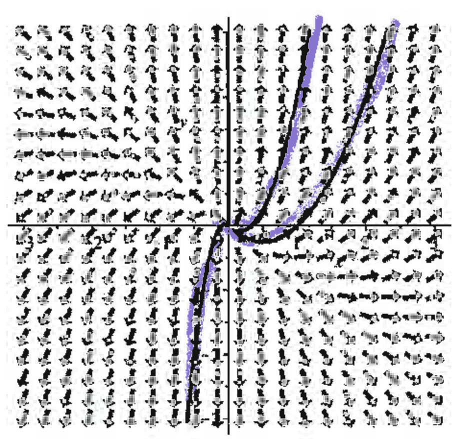
$(x_1(t), x_2(t)) = (2,2)$ is an unstable saddle.

$(x_1(t), x_2(t)) = (-2,-2)$ is an asympt. stable spiral.

basin of attraction:



23.)

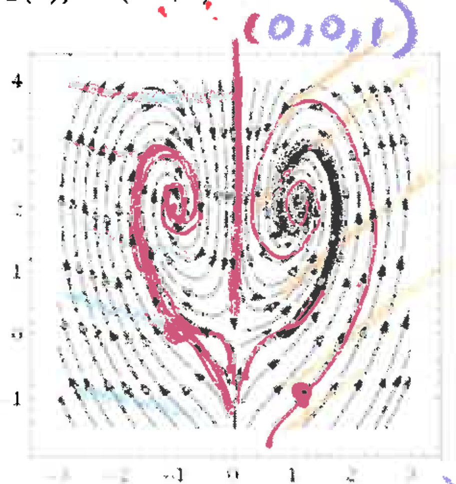


$(x_1(t), x_2(t)) = (0,0)$ is an unstable node.

No basin of attraction: $x_1 < 0$.

Problems 21-23 show the stream plot in the $x_1 - x_2$-plane for a system of two first order differential equations In addition to determining and classifying all equilibrium solutions, also draw the trajectories satisfying the initial values $(x_1(0), x_2(0)) = (0,1)$, $(x_1(0), x_2(0)) = (1,0)$, $(x_1(0), x_2(0)) = (1,2)$, $(x_1(0), x_2(0)) = (-1,0)$. Also describe the basins of attraction.

21.)



$(x_1(t), x_2(t)) = (0,0)$ is an unstable saddle.

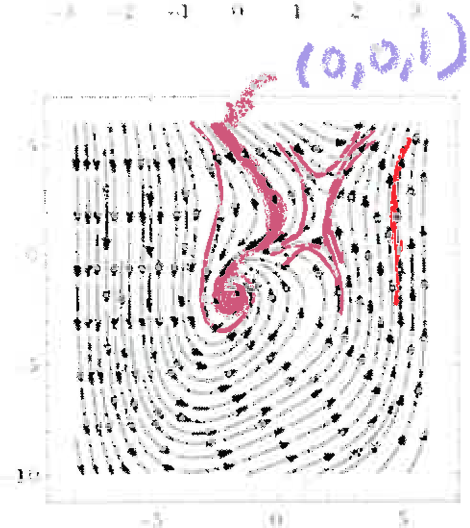
$(x_1(t), x_2(t)) = (1,2)$ is an asymptotically stable node.

basin of attraction: $x_1 > 0$.

$(x_1(t), x_2(t)) = (-1,2)$ is an asymptotically stable node.

basin of attraction: $x_1 < 0$.

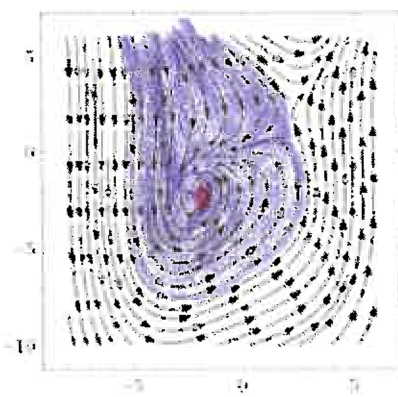
22.)



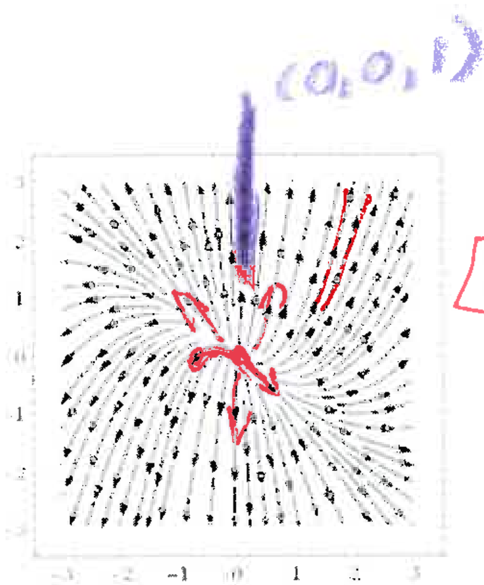
$(x_1(t), x_2(t)) = (2,2)$ is an unstable saddle.

$(x_1(t), x_2(t)) = (-2,-2)$ is an asympt. stable spiral.

basin of attraction:



23.)



$(x_1(t), x_2(t)) = (0,0)$ is an unstable node.

No basin of attraction: $\cancel{x_1 < 0}$.

3.6 Variation of Parameters

Solve $y'' - 2y' + y = e^t ln(t)$

1) Find homogeneous solutions: Solve $y'' - 2y' + y = 0$

Guess: $y = e^{rt}$, then $y' = re^{rt}$, $y'' = r^2 e^{rt}$, and

$$r^2 e^{rt} - 2re^{rt} + e^{rt} = 0 \text{ implies } r^2 - 2r + 1 = 0$$

$(r-1)^2 = 0$, and hence $r = 1$

General homogeneous solution: $y = c_1 e^t + c_2 te^t$

since have two linearly independent solutions: $\{e^t, te^t\}$

2.) Find a non-homogeneous solution:

Sect. 3.5 method: Educated guess

Sect. 3.6: **Guess** $y = u_1(t)e^t + u_2(t)te^t$ **and solve for** u_1 **and** u_2

Only partial credit if you use 3.5 method

$$u_1(t) = \int \frac{\begin{vmatrix} 0 & \phi_2 \\ 1 & \phi_2' \end{vmatrix}}{\begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}} g(t)dt = -\int \frac{\phi_2(t)g(t)}{W(\phi_1,\phi_2)}dt = -\int \frac{(te^t)(e^t ln(t))}{e^{2t}}dt$$

$$= -\int tln(t) = -[\frac{t^2 ln(t)}{2} - \int \frac{t}{2}] = -\frac{t^2 ln(t)}{2} + \frac{t^2}{4}$$

$$u_2(t) = \int \frac{\begin{vmatrix} \phi_1 & 0 \\ \phi_1' & 1 \end{vmatrix}}{\begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}} g(t)dt = \int \frac{\phi_1(t)g(t)}{W(\phi_1,\phi_2)}dt = \int \frac{(e^t)(e^t ln(t))}{e^{2t}}dt$$

$\checkmark-$

$$= \int ln(t) = tln(t) - t$$

$$W(\phi_1, \phi_2) = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix}$$

$u = ln(t)$ $\quad dv = tdt$

$du = \frac{dt}{t}$ $\quad v = \frac{t^2}{2}$

$u = ln(t)$ $\quad dv = dt$

$du = \frac{dt}{t}$ $\quad v = t$

General solution : $y = c_1 e^t + c_2 te^t + (-\frac{t^2 ln(t)}{2} + \frac{t^2}{4})e^t + (tln(t) - t)te^t$

which simplifies to $y = c_1 e^t + c_2 te^t + (\frac{ln(t)}{2} - \frac{3}{4})t^2 e^t$

Sect.3.6: **Guess** $y = u_1(t)e^t + u_2(t)te^t$ **and solve for** u_1 **and** u_2

$y' = u_1'e^t + u_1e^t + u_2'te^t + u_2(e^t + te^t) = e^{2t} + te^{2t} - te^{2t} - e^{2t}$.

Two unknown functions, u_1 and u_2, but only one equation ($y'' - 2y' + y = e^t ln(t)$). Thus might be OK to choose 2nd eq'n.

Avoid 2nd derivative in y'': **Choose** $u_1'e^t + u_2'te^t = 0$

Hence $y' = u_1e^t + u_2(e^t + te^t)$.

and $y'' = u_1'e^t + u_1e^t + u_2'(e^t + te^t) + u_2(e^t + e^t + te^t)$.

$$= u_1'e^t + u_1e^t + u_2'e^t + u_2'te^t + u_2(2e^t + te^t).$$

$$= u_1e^t + u_2'e^t + u_2(2e^t + te^t).$$

$$\boxed{\text{Solve } y'' - 2y' + y = e^t ln(t)}$$

$$u_1e^t + u_2'e^t + u_2(2e^t + te^t) - 2[u_1e^t + u_2(e^t + te^t)] + u_1e^t + u_2te^t = e^t ln(t)$$

$$u_2'e^t + 2u_2e^t + u_2te^t - 2u_2e^t - 2u_2te^t + u_2te^t = e^t ln(t)$$

$u_2' = ln(t)$ or in other words, $\frac{du_2}{dt} = ln(t)$

Thus $\int du_2 = \int ln(t)dt$

$u_2 = tln(t) - t$. Note only need one solution, so don't need $+C$.

$y = u_1(t)e^t + [tln(t) - t]te^t$

$u_1'e^t + u_2'te^t = 0$. Thus $u_1' + u_2't = 0$. Hence $u_1' = -u_2't = -tln(t)$

Thus $u_1 = -\int tln(t)dt = -\frac{t^2 ln(t)}{2} + \frac{t^2}{4}$

Thus the general solution is

$$y = c_1e^t + c_2te^t + \left(-\frac{t^2 ln(t)}{2} + \frac{t^2}{4}\right)e^t + (tln(t) - t)te^t$$