9.2

$mr'' = \frac{-GMm}{r^2}$

Let $v = r'$, then $v' = r''$

Thus we obtain system of non-linear equations:

$$dr/dt = r' = v$$
$$dv/dt = v' = \frac{-GM}{r^2}$$

← autonomous no t term on RHS

Note $v' = \frac{-GM}{r^2}$ involves 3 variables: v, t, r

Eliminate t: $v' = \frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = \frac{dv}{dr}v$

Thus $mv' = \frac{-GMm}{r^2}$ becomes $m\frac{dv}{dr}v = \frac{-GMm}{r^2}$

Separate variables: $\int mdvv = \int \frac{-GMm}{r^2}dr$

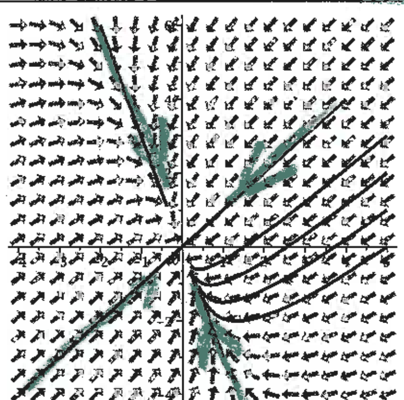
$$\frac{1}{2}mv^2 = \frac{GMm}{r} + E \text{ where } E \text{ is a constant.}$$

Thus we have derived the physics formula, conservation of energy:

$$\frac{1}{2}mv^2 + \frac{-GMm}{r} = E$$

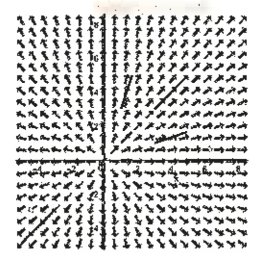
I.e., Kinetic Energy + Potential Energy = constant

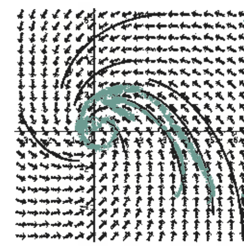
$x' = -4x - y$
$y' = -3x + 2y$



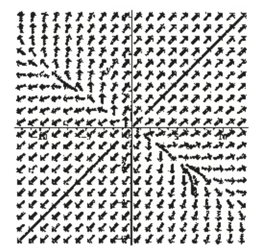
2 neg real evalue
asym stable
node

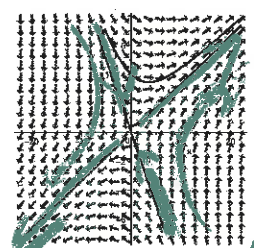
Suppose the following represent direction fields of linear systems of 1rst order differential equations in the phase plane. What can you say about solutions to these systems of equations.





Spiral
asym stable
approaching equilibrium soln
complex $r + bi$
$r < 0$

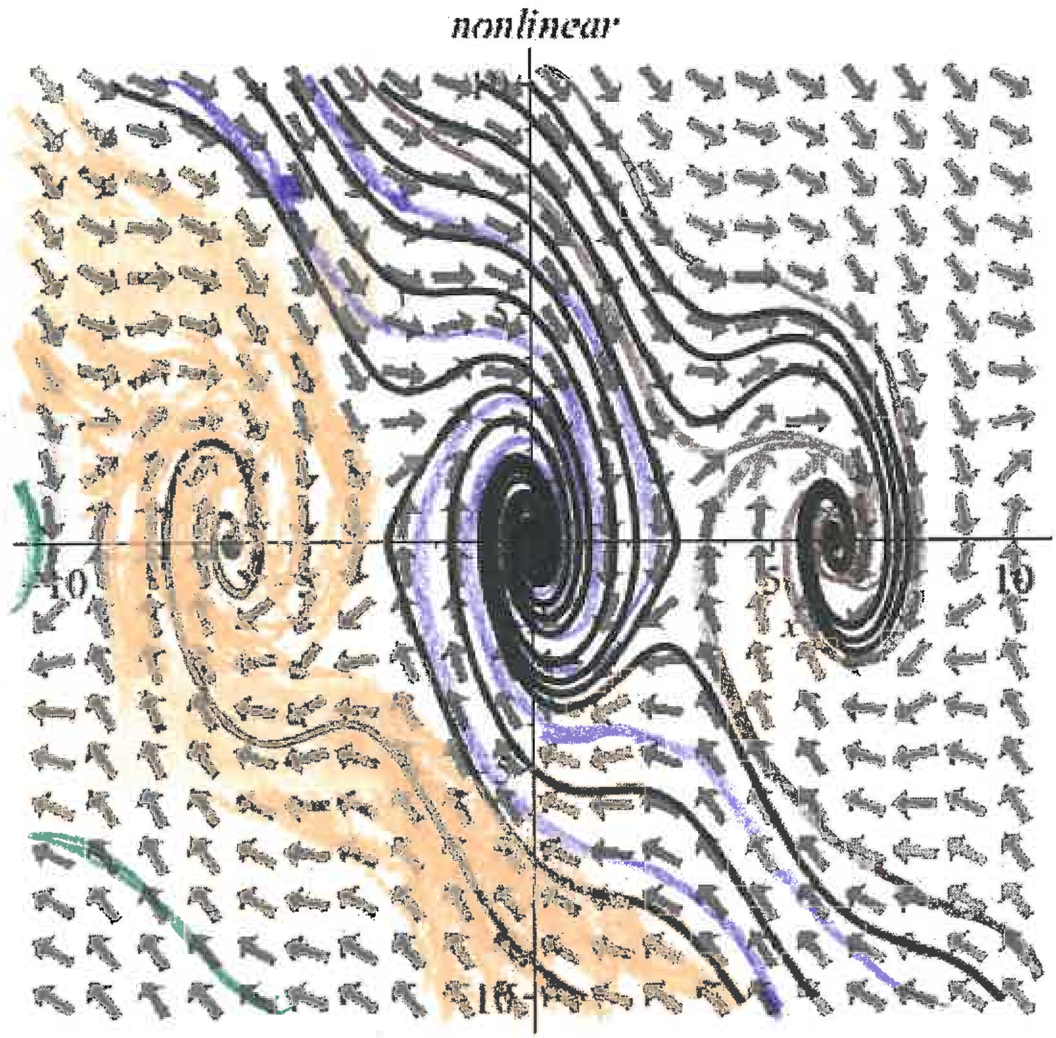




saddle
unstable

$$x' = y$$
$$y' = -y - 6\sin(x)$$

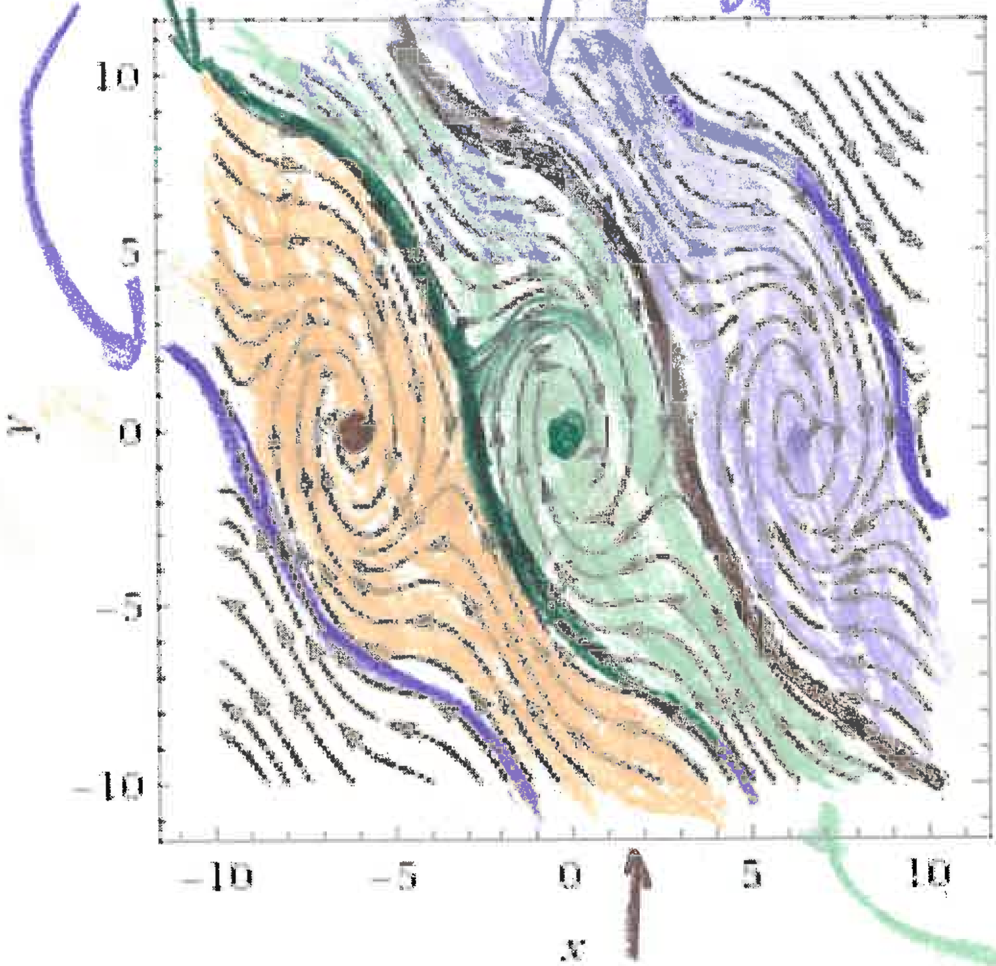
```
> DEplot([diff(x(t), t) = y(t), diff(y(t), t) =-y(t) - 6 sin(x(t))], [x(t), y(t)], t=-2..9, x=-10
    ..10, y=-10..10, {[0,-4,-4], [0,-1, 4], [0, 4,-4], [0, 5, 5], [0, 0, 5], [0, 0, -5], [0, 0,
    6], [0, 0, 7], [0, 0, -7], [0, 3, 3], [0, 4, 4]}, arrows = THICK, stepsize = .01, linecolor
    = black, title = `nonlinear`)
```



Input interpretation:

stream plot	$\{y, -y - 6\sin(x)\}$	$x = -10$ to
		$y = -10$ to

Plot:



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Example: $x' = x - xy, \; y' = -y + xy$ Eqn (*)

critcal points: $x' = x - xy = x(1-y) = 0$ implies $x = 0$ or $y = 1$,
$y' = -y + xy = y(-1+x) = 0$ implies $y = 0$ or $x = 1$. Break into cases using 1rst equation (or any equation of your choice)

Case 1: If $x = 0$, then $y = 0$ by second case. Thus (0, 0) is a critical point.
Case 2: If $y = 1$, then $x = 1$ by second case. Thus (1, 1) is a critical point.

Jacobian matrix: $\begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} 1-y & -x \\ y & -1+x \end{bmatrix}$

Case 1: $(x_0, y_0) = (0, 0)$.

The linear approximation to non-linear differential equation (*) is

$$\mathbf{x}' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{x}$$

Determine stability of critical point (0, 0) of linear approximation:

Short method: *eigenvalues* $= 1, -1$, thus (0, 0) is an unstable saddle point of the linear approximation. If we slightly perturb $(p, q) = (a+d, ad-bc) = (0, -1)$, we still have an unstable saddle point.

Thus $(x_0, y_0) = (0, 0)$ is an unstable saddle point of the nonlinear system of D.E.,
$x' = x - xy, \; y' = -y + xy$

Longer method: $det(A - rI) = \begin{vmatrix} 1-r & 0 \\ 0 & -1-r \end{vmatrix} = (1-r)(-1-r) = r^2 - 1 = 0$

Thus $r = \frac{0 \pm \sqrt{0^2 - 4(-1)}}{2} = \frac{p \pm \sqrt{p^2 - 4(q)}}{2}$ where $r^2 - pr + q = 0$. I.e., $p = 0$, $q = -1$. See figure 9.1.9 in your text.

Thus (0, 0) is an unstable saddle point of the linear approximation. If we slightly perturb the linear approximation differential equation, then we will still have one positive and one negative eigenvalue (since values close to the eigenvalue 1 will still be positive and values close to the eigenvalue -1 will still be negative. Thus we still have an unstable saddle point. Similarly, in figure 9.1.9 when $p = 0$, $q = -1$, we get an unstable saddle point for small perturbations of p and q.

Thus $(x_0, y_0) = (0, 0)$ is an unstable saddle point of nonlinear system of D.E.,
$$x' = x - xy, \; y' = -y + xy$$

Case 2: $(x_0, y_0) = (1, 1)$.

The linear translated approximation to non-linear differential equation (*) is

$$\mathbf{x}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x}$$

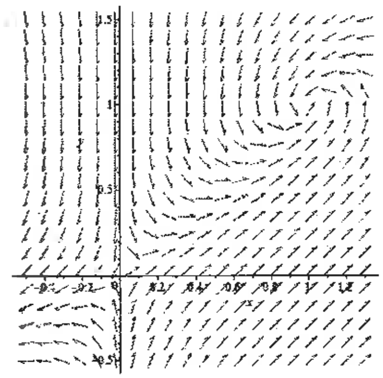
$det(A - rI) = \begin{vmatrix} -r & -1 \\ 1 & -r \end{vmatrix} = r^2 + 1 = 0$. Thus $r = \frac{0 \pm \sqrt{0^2 - 4(1)}}{2} = \pm i$.

Thus (0, 0) is a stable center point of the linear translated approximation.

If we take complex numbers, $a \pm bi$ close to the eigenvalues $r = \pm i$, then b will be close to 1 and thus positive, but a will be close to 0 and thus could be positive or negative or 0.

Alternatively, see figure 9.1.9 in your text where $\boldsymbol{p = a + d = 0}$ and $\boldsymbol{q = ad - bc = 1}$.

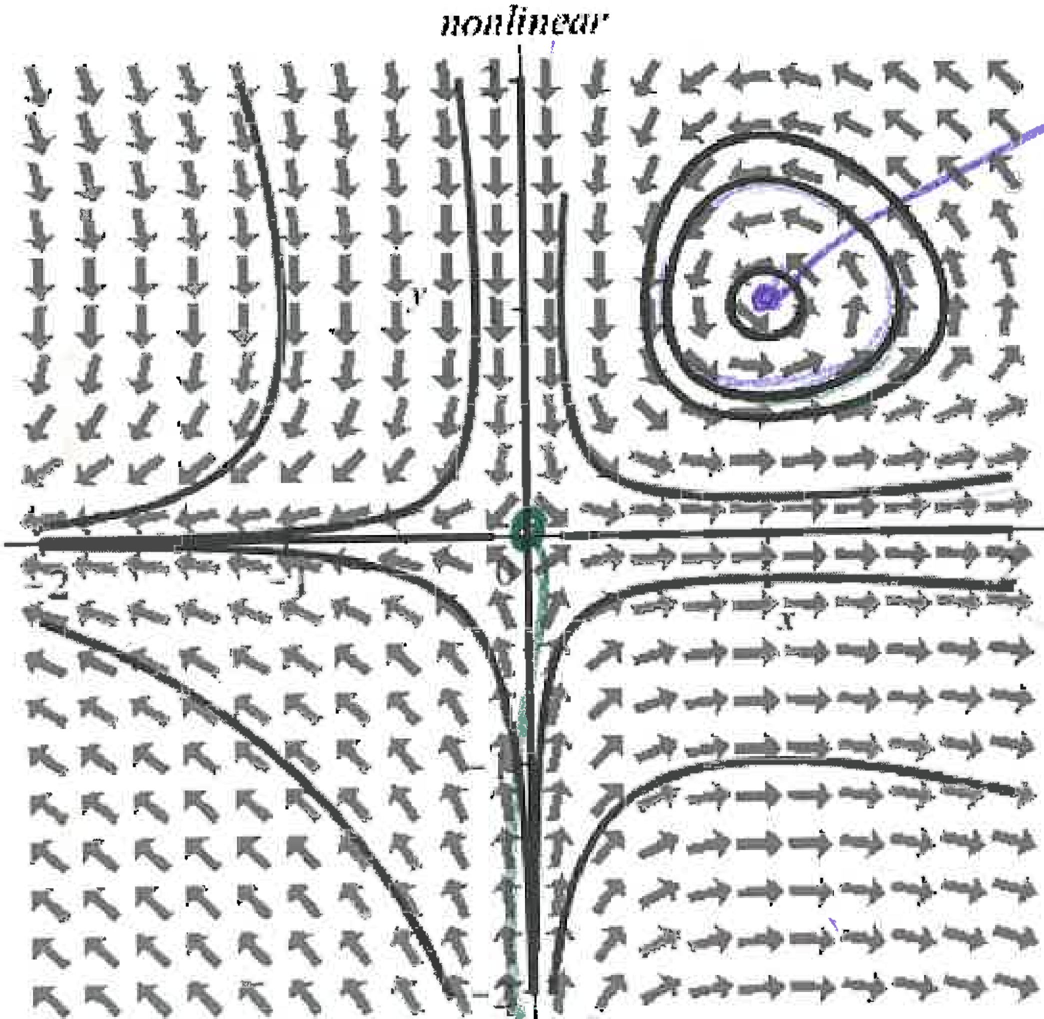
Hence for the nonlinear equation (*), the critical point (1, 1) is one of the following
1.) stable center point
2.) unstable spiral point
3.) asymptotically stable spiral point.



wrong graph

$x' = x - xy$
$y' = -y + xy$

```
with(DEtools) :
>
DEplot( [diff(x(t), t) = x(t) - x(t)·y(t), diff(y(t), t) = -y(t) + x(t)·y(t) ], [x(t), y(t) ], t = -2 ..9, x =
  -2 ..2, y = -2 ..2, {[0, .7, .7], [0, .9, .9], [0, 1.5, 1.5], [0, .3, .3], [0, -1, -1], [0, -1, 1], [0, 1, -1],
  [0, 1, 0], [0, -1, 0], [0, 0, 1], [0, 0, -1], [0, -.3, -.3], [0, -.3, .3], [0, .3, -.3]}, arrows = THICK,
  stepsize = .01, linecolor = black, title = `nonlinear`);
```

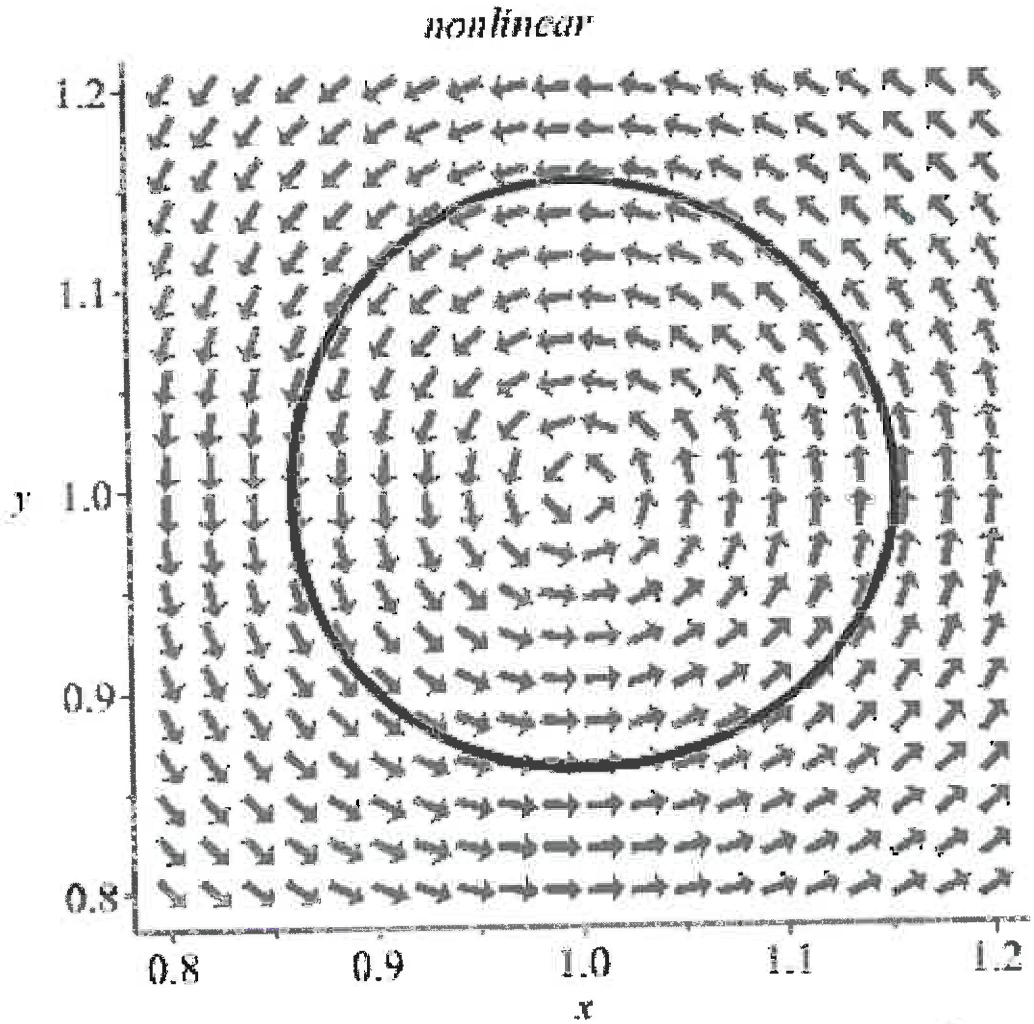


stable critical pt center

unstable
stable

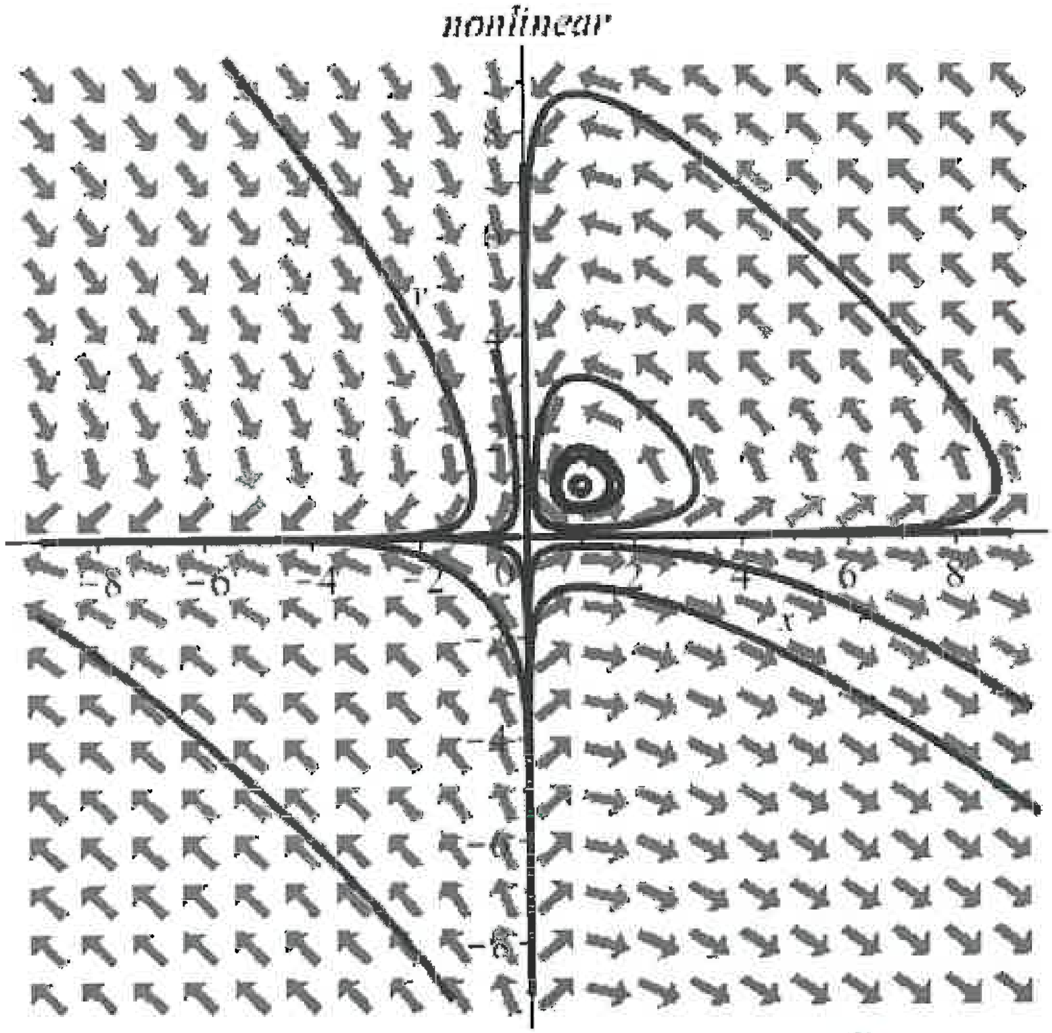
No asym stable critical pt
$\Rightarrow$ no basin of attraction

> *DEplot*([*diff* (*x*(*t*), *t*) = *x*(*t*) - *x*(*t*) · *y*(*t*), *diff* (*y*(*t*), *t*) =-*y*(*t*) + *x*(*t*) · *y*(*t*)], [*x*(*t*), *y*(*t*)], t=-2 ..9, x = 0.8 ..1.2, y = 0.8 ..1.2, { [0, .7, .7], [0, .9, .9], [0, 1.5, 1.5], [0, .3, .3], [0,-1,-1], [0, -1, 1], [0, 1,-1], [0, 1, 0], [0,-1, 0], [0, 0, 1], [0, 0,-1], [0,-.3,-.3], [0,-.3, .3], [0, .3, -.3] }, *arrows* = *THICK*, *stepsize* = .01, *linecolor* = *black*, *title* = `*nonlinear*`);



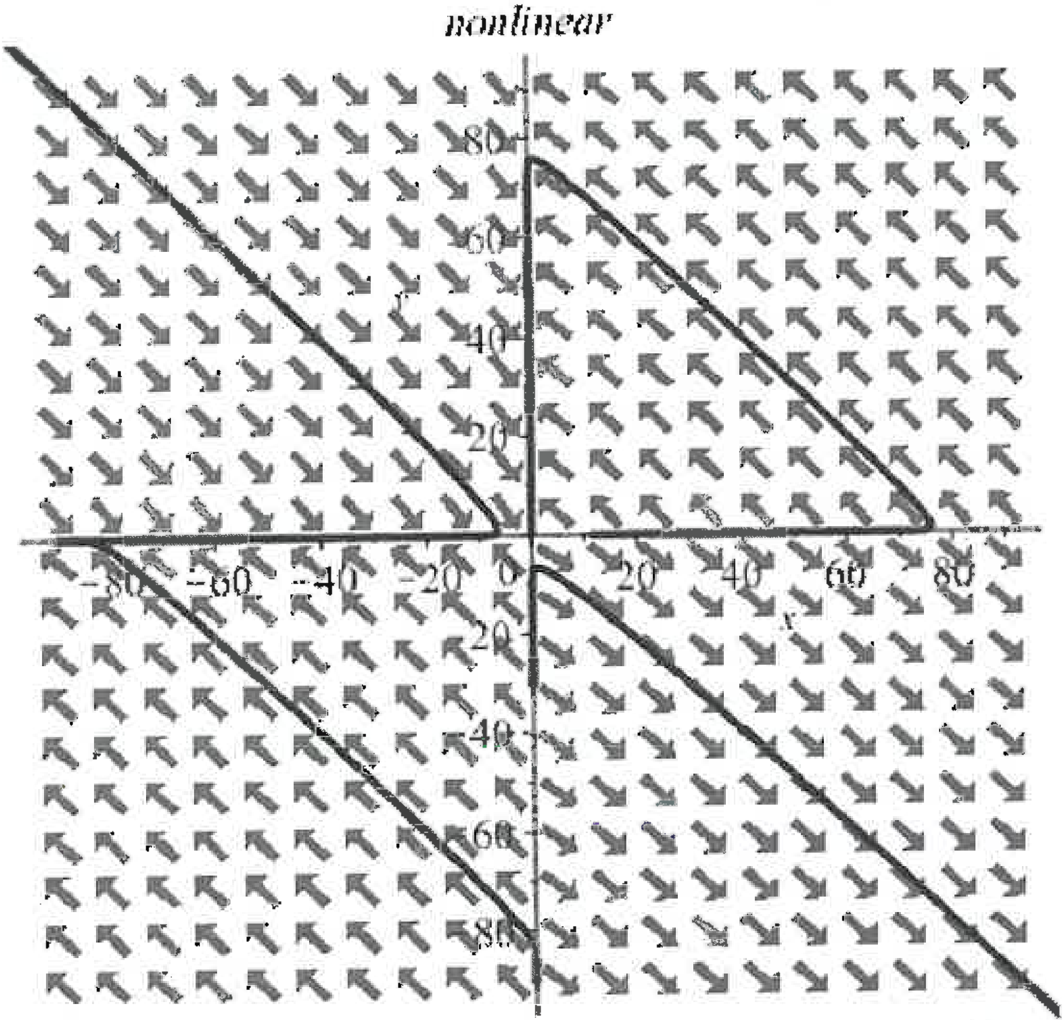
zoom in

```
DEplot( [diff(x(t), t) = x(t) - x(t)·y(t), diff(y(t), t) = -y(t) + x(t)·y(t)], [x(t), y(t)], t = -2..9, x =
    -9..9, y = -9..9, {[0, .7, .7], [0, .9, .9], [0, 1.5, 1.5], [0, .3, .3], [0, -1, -1], [0, -1, 1], [0, 1, -1],
    [0, 1, 0], [0, -1, 0], [0, 0, 1], [0, 0, -1], [0, -.3, -.3], [0, -.3, .3], [0, .3, -.3], [0, 5.5, 5.5], [0, -5,
    -5]}, arrows = THICK, stepsize = .01, linecolor = black, title = `nonlinear`);
```



zoom out

DEplot([*diff*($x(t)$, t) $= x(t) - x(t) \cdot y(t)$, *diff*($y(t)$, t) $= -y(t) + x(t) \cdot y(t)$], [$x(t)$, $y(t)$], t = -2 ..9, x = -90 ..90, y = -90 ..90, { [0, -40, -40], [0, -40, 40], [0, 40, -40], [0, 40, 40]}, *arrows* = *THICK*, *stepsize* = .01, *linecolor* = *black*, *title* = \`*nonlinear*\`);



zoom far out

7.6: Complex eigenvalue example: Solve $\mathbf{x}' = \begin{bmatrix} 3 & -13 \\ 5 & 1 \end{bmatrix} \mathbf{x}$

Step 1 Find eigenvalues: $det(A - rI) = 0$

$$det(A - rI) = \begin{vmatrix} 3-r & -13 \\ 5 & 1-r \end{vmatrix} = (3-r)(1-r) + 65 = r^2 - 4r + 68 = 0$$

Thus $r = \frac{4 \pm \sqrt{4^2 - 4(68)}}{2} = \frac{4 \pm \sqrt{4(4-68)}}{2} = \frac{4 \pm 2\sqrt{-64}}{2} = 2 \pm 8i$

spiral

$2 > 0$

unstable

Step 2 Find eigenvectors: Solve $(A - rI)\mathbf{x} = \mathbf{0}$

$$A - (2 \pm 8i)I = \begin{bmatrix} 3-(2 \pm 8i) & -13 \\ 5 & 1-(2 \pm 8i) \end{bmatrix} = \begin{bmatrix} 1 \mp 8i & -13 \\ 5 & -1 \mp 8i \end{bmatrix}$$

Solve $\begin{bmatrix} 1 \mp 8i & -13 \\ 5 & -1 \mp 8i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 \mp 8i & -13 \\ 5 & -1 \mp 8i \end{bmatrix} \begin{bmatrix} 13 \\ 1 \mp 8i \end{bmatrix} = \begin{bmatrix} (1 \mp 8i)13 - 13(1 \mp 8i) \\ 5(13) + (-1 \mp 8i)(1 \mp 8i) \end{bmatrix} = \begin{bmatrix} 0 \\ 65 + (-1 + 64i^2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus any non-zero multiple of $\begin{bmatrix} 13 \\ 1 \mp 8i \end{bmatrix}$ is an eigenvector of A with eigen value $2 \pm 8i$.

Note: $\begin{bmatrix} 1 \pm 8i \\ 5 \end{bmatrix}$ is a multiple of $\begin{bmatrix} 13 \\ 1 \mp 8i \end{bmatrix}$ since $\begin{bmatrix} 1 \mp 8i & -13 \\ 5 & -1 \mp 8i \end{bmatrix} \begin{bmatrix} 1 \pm 8i \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Thus we can use either $\begin{bmatrix} 1 \pm 8i \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \pm i \begin{bmatrix} 8 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 13 \\ 1 \mp 8i \end{bmatrix}$ or any nonzero multiple.

General solution:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{2t} \left(\begin{bmatrix} 1 \\ 5 \end{bmatrix} cos(8t) - \begin{bmatrix} 8 \\ 0 \end{bmatrix} sin(8t) \right) + c_2 e^{2t} \left(\begin{bmatrix} 1 \\ 5 \end{bmatrix} sin(8t) + \begin{bmatrix} 8 \\ 0 \end{bmatrix} cos(8t) \right)$$

Slope field for x_2 vs x_1: $\frac{dx_2}{dx_1} = \frac{\frac{dx_2}{dt}}{\frac{dx_1}{dt}} = \frac{x_2'}{x_1'} = \frac{3x_1 - 13x_2}{5x_1 + x_2}$

Note slope 0's occur when $3x_1 - 13x_2 = 0$, ie, $x_2 = \frac{13}{3}x_1$.

Note slope ∞'s occur when $5x_1 + x_2 = 0$, ie, $x_2 = -5x_1$.

Determine where slopes are positive vs negative for regions between these lines.

For example, along the x_2 axis slope is negative: $x_1 = 0$ and $\frac{dx_2}{dx_1} = \frac{-13x_2}{x_2} = -13$

For example, along the x_1 axis slope is positive: $x_2 = 0$ and $\frac{dx_2}{dx_1} = \frac{3x_1}{5x_1} = \frac{3}{5}$