Equilibrium soln   constant soln

$\vec{X}' = A\vec{X}$

$\vec{X}' = 0 \ \forall t \Rightarrow \vec{X} = 0$

$x_1(t) = 0$

$x_2(t) = 0$

Ch 7 and 9

Suppose an object moves in the 2D plane (the x_1, x_2 plane) so that it is at the point $(x_1(t), x_2(t))$ at time t. Suppose the object's velocity is given by

$$x_1'(t) = ax_1 + bx_2,$$
$$x_2'(t) = cx_1 + dx_2$$

Or in matrix form $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

To solve, find eigenvalues and corresponding eigenvectors:

$$\begin{vmatrix} a-r & b \\ c & d-r \end{vmatrix} = (a-r)(d-r) - bc = r^2 - (a+d)r + ad - bc = 0.$$

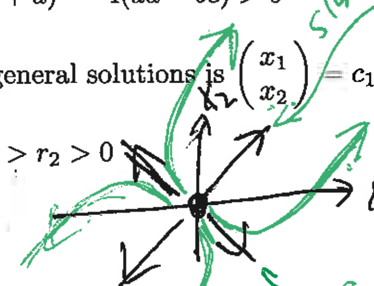
Thus $r = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$

Case 1: $(a+d)^2 - 4(ad-bc) > 0$

slope $\frac{v_2}{v_1}$ $r_1 > r_2 > 0$

Hence the general solutions is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{r_1 t} + c_2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} e^{r_2 t}$

Case 1a: $r_1 > r_2 > 0$

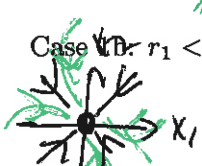


unstable

r_2 e. value $< r_1$

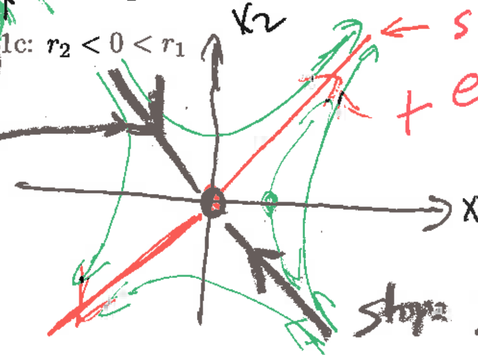
slope $\frac{w_2}{w_1}$

Case 1b: $r_1 < r_2 < 0$



stable

Case 1c: $r_2 < 0 < r_1$



- e. value

slope $\frac{v_2}{v_1}$

+ eigenvalue

$c_2 = 0$

$x_1 = c_1 v_1 e^{r_1 t}$

$x_2 = c_1 v_2 e^{r_1 t}$

$x_2 = \frac{v_2}{v_1} x_1$

slope $\frac{w_2}{w_1}$

unstable saddle pt

Case 2: $(a+d)^2 - 4(ad-bc) = 0$

Case 2i: Two independent eigenvectors:

The general solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{rt} + c_2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} e^{rt}$

Case 2ii: One independent eigenvectors:

The general solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{rt} + c_2 \left[\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} t + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right] e^{rt}$

$r > 0$ $r < 0$

Case 2a: $r > 0$ Case 2b: $r < 0$

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Case 3: $(a+d)^2 - 4(ad-bc) < 0$. I.e., $r = \lambda \pm i\mu$

Suppose eigenvector corresponding to eigenvalue is

$$\begin{pmatrix} v_1 \pm iw_1 \\ v_2 \pm iw_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \pm i \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

Then general solution is

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{\lambda t} \left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} cos(\mu t) - \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} sin(\mu t) \right) + c_2 e^{\lambda t} \left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} sin(\mu t) + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} cos(\mu t) \right)}$$

Case 3a: $\lambda > 0$

Case 3a: $\lambda < 0$

Case 3a: $\lambda = 0$