7.4 - 7.6, 9.1

Solve the homogeneous linear DE: $\mathbf{x}' - A\mathbf{x} = \mathbf{0}$

$\mathbf{x}' = A\mathbf{x}$ $\quad$ $\boxed{\text{Guess } x = \mathbf{v}e^{rt}.}$ $\quad$ Plug in to find $\mathbf{v}$ and r:

$[\mathbf{v}e^{rt}]' = A\mathbf{v}e^{rt}$ implies $r\mathbf{v}e^{rt} = A\mathbf{v}e^{rt}$ implies $r\mathbf{v} = A\mathbf{v}$.

$\boxed{\text{Thus } \mathbf{v} \text{ is an eigenvector with eigenvalue } r.}$

Note since the equation is homogeneous and linear,

linear combinations of solutions are also solutions:

Suppose $\mathbf{x} = \mathbf{f_1}(t)$ and $\mathbf{x} = \mathbf{f_2}(t)$ are solutions to $\mathbf{x}' = A\mathbf{x}$.

Then $\mathbf{f_1}' = A\mathbf{f_1}$ and $\mathbf{f_2}' = A\mathbf{f_2}$

Thus $[c_1\mathbf{f_1} + c_2\mathbf{f_2}]' = c_1\mathbf{f_1}' + c_2\mathbf{f_2}' = c_1A\mathbf{f_1} + c_2A\mathbf{f_2} = A(c_1\mathbf{f_1} + c_2\mathbf{f_2})$.

Suppose an object moves in the 2D plane (the x_1, x_2 plane) so that it is at the point $(x_1(t), x_2(t))$ at time t. Suppose the object's velocity is given by

$$x_1'(t) = 4x_1 + x_2,$$
$$x_2'(t) = 5x_1$$

Or in matrix form $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 4 & 1 \\ 5 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

To solve, find eigenvalues and corresponding eigenvectors:

$$\begin{vmatrix} 4-r & 1 \\ 5 & -r \end{vmatrix} = (4-r)(-r) - 5 = r^2 - 4r - 5 = (r-5)(r+1).$$

Thus $r = -1, 5$ are eigenvalues.

$$\begin{pmatrix} 5 & 1 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eigenvectors associated to eigenvalue $r = -1$: $\begin{pmatrix} 5 & 1 \\ 5 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & \frac{1}{5} \\ 0 & 0 \end{pmatrix}$

Thus x_2 is free and $x_1 + \frac{1}{5}x_2 = 0$

Hence the eigenspace corresponding to $r = -1$ is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{5}x_2 \\ x_2 \end{pmatrix} = x_2\begin{pmatrix} -\frac{1}{5} \\ 1 \end{pmatrix}$$

Thus $\begin{pmatrix} -1 \\ 5 \end{pmatrix}$ is an eigenvector with eigenvalue $r = -1$

Hence $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} e^{-t}$ is a solution.

E. vectors associated to e. value $r = 5$: $\begin{pmatrix} -1 & 1 \\ 5 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Thus $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue $r = 5$ since it is a nonzero solution to the above equation.

Hence $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t}$ is also a solution.

Hence the general solutions is $\boxed{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1\begin{pmatrix} -1 \\ 5 \end{pmatrix} e^{-t} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{5t}}$

Or in non-matrix form: $x_1(t) = -c_1e^{-t} + c_2e^{5t}$
$x_2(t) = 5c_1e^{-t} + c_2e^{5t}$