$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$ no solution

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$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Check:
$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \& \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$



See handouts

Compare to solving linear homogeneous differential eqn:

Ex: $ay'' + by' + cy = g(t)$

1.) Easily solve homogeneous DE: $ay'' + by' + cy = 0$

$y = e^{rt} \Rightarrow ar^2 + br + c = 0 \Rightarrow y = c_1\phi_1 + c_2\phi_2$ for homogeneous solution (see sections 3.1, 3.3, 3.4).

2.) More work: Find one solution to $ay'' + by' + cy = g(t)$ (see sections 3.5, 3.6)

If $y = \psi(t)$ is a soln, then general soln to $ay''+by'+cy = g(t)$ is

$$\boxed{y = c_1\phi_1 + c_2\phi_2 + \psi}$$

homog + non homog

Check: $a\phi_1'' + b\phi_1' + c\phi_1 = 0$

$a\phi_2'' + b\phi_2' + c\phi_2 = 0$

$a\psi'' + b\psi' + c\psi = g(t)$

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To solve $ay'' + by' + cy = g_1(t) + g_2(t)$ $+ g_3$

1.) Solve $ay'' + by' + cy = 0 \Rightarrow y = c_1\phi_1 + c_2\phi_2$ for homogeneous solution.

2a.) Solve $ay'' + by' + cy = g_1(t) \Rightarrow y = \psi_1$

2b.) Solve $ay'' + by' + cy = g_2(t) \Rightarrow y = \psi_2$

2c.

General solution to $ay'' + by' + cy = g_1(t) + g_2(t)$ is

$$\boxed{y = \underbrace{c_1\phi_1 + c_2\phi_2}_{\text{homog}} + \underbrace{\psi_1 + \psi_2}_{\text{nonhomog}}}$$

Exam 2 review:

**To solve a single differential equation, for exam 2, use Ch 5 methods:**

A.) If you have an $\boxed{\text{Euler equation, } x^2y'' + \alpha xy' + \beta y = 0}$ where $\alpha, \beta$ are constants, use simple 5.4 method (guess $y = |x|^r$, breaks into standard 3 cases, see 5.4 handouts ). plug in $y = x^r$

B.) Suppose you are interested in the solution near $x = x_0$, then we can find

(1.) exact solution solution by solving for the series solution (ex: see 5.2 handout) $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ where $a_n = f(a_0, a_1)$

not an Euler eqn

(2.) An approximate solution by determining the first few terms in the series solution (ex: see 5.5 part 2 handout) $y = (a_0 + a_1x + a_2x^2 + \cdots + a_kx^k)$ $x^r$ degree $k$ approx

see also 5.5 ex

Determine if $x_0$ is an ordinary point, regular singular value, or irregular singular value. $x_0 = 0$

If $x_0$ is an ordinary point, solution near $x_0$ is $\Sigma_{n=0}^{\infty} a_n(x - x_0)^n$.

If $x_0$ is a regular singular point, solution near $x_0$ is $\Sigma_{n=0}^{\infty} a_n(x - x_0)^{n+r}$.

When (and where) do you know when solution exists?

What are the subparts of these problems?

Look at theory including existence, uniqueness, domain of solution, linearity.

Find approx soln $y = a_0 x^r + a_1 x^{r+1} + a_2 x^{r+2} + \dots a_k x^{r+k}$

**5.5:** ~~Solve~~ $x^2y'' - x(2+x)y' + (2+x^2)y = 0$

$x^2 \qquad x^2 \qquad x^2$

$$\underbrace{y = x^r\left(\sum_{i=0}^{k} a_i x^i\right)}_{\text{approx soln}}$$

$p(x) = -\frac{x(2+x)}{x^2} = -\frac{2+x}{x}$. Thus $x_0 = 0$ is a singular value.

$q(x) = \frac{2+x^2}{x^2}$ also implies $x_0 = 0$ is a singular value.

$xp(x) = -(2+x)$ and $x^2q(x) = 2+x^2$. Thus $x_0 = 0$ is a regular singular value.

Suppose $\boxed{y = \sum_{n=0}^{\infty} a_n x^{n+r}}$ is a solution. WLOG assume $a_0 \neq 0$ (otherwise one can reindex the summation).

Then $y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$ and $y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$

$x^2y'' - x(2+x)y' + (2+x^2)y$

$= x^2 \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} - (2x + x^2)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + (2+x^2)\sum_{n=0}^{\infty} a_n x^{n+r}$

$= \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} - \sum_{n=0}^{\infty} 2(n+r)a_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)a_n x^{n+r+1}$
$\quad + \sum_{n=0}^{\infty} 2a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2}$

reindex — reindex

$= \sum_{n=0}^{\infty}[(n+r)(n+r-1) - 2(n+r) + 2]a_n x^{n+r} - \sum_{n=1}^{\infty}(n+r-1)a_{n-1}x^{n+r} + \sum_{n=2}^{\infty} a_{n-2}x^{n+r}$

write separately $n=0$ & $n=1$ term — $n=1$

$= [r(r-1) - 2r + 2]a_0 x^r + [(1+r)r - 2(1+r) + 2]a_1 x^{r+1} - ra_0 x^{r+1}$
$\quad + \sum_{n=2}^{\infty}[(n+r)(n+r-1) - 2(n+r) + 2]a_n x^{n+r} - \sum_{n=2}^{\infty}(n+r-1)a_{n-1}x^{n+r} + \sum_{n=2}^{\infty} a_{n-2}x^{n+r}$

$= [r(r-1) - 2r + 2]a_0 x^r + ([(1+r)r - 2(1+r) + 2]a_1 - ra_0)x^{r+1}$
$\quad + \sum_{n=2}^{\infty}\left([(n+r)(n+r-1) - 2(n+r) + 2]a_n - (n+r-1)a_{n-1} + a_{n-2}\right)x^{n+r}$

simplify algebra

$= [r^2 - r - 2r + 2]a_0 x^r + ([r + r^2 - 2 - 2r + 2]a_1 - ra_0)x^{r+1}$
$\quad + \sum_{n=2}^{\infty}([(n+r)(n+r-3) + 2]a_n - (n+r-1)a_{n-1} + a_{n-2})x^{n+r}$

$= \overbrace{[r^2 - 3r + 2]}^{=0}a_0 x^r + (\overbrace{[r^2 - r]a_1 - ra_0}^{=0})x^{r+1}$
$\quad + \sum_{n=2}^{\infty}\underbrace{([n^2 + 2rn + r^2 - 3n - 3r + 2]a_n - (n+r-1)a_{n-1} + a_{n-2})}_{=0}x^{n+r} = 0$

$= \sum 0 x^{n+r}$